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The Optimality Conditions for Cone-Preinvex Set-Valued Functions*

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Abstract: This paper deals with the minimization problems of cone-preinvex set-valued functions in the topological vector space. The optimality conditions for vector optimization of cone-preinvex set-valued functions are obtained.

Key words: cone-preinvex set-valued functions; optimality conditions; weak efficient solution

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Many results of the optimality conditions for set-valued functions have been obtained in recent years, for example, Li[1], ZhongFei Li and GuangYa Chen[2], etc. The notion of preinvex for scalar-valued functions was introduced into literature by Weir[3] and Weir[4] by relaxing the convexity assumption on the domain set of the functions. Davinder Bhatia[5] had extended the class of cone-convex set-valued functions to the class of cone-preinvex set-valued functions. A fractional programming problem involving set-valued functions has been considered.

Motivated by Li[1], in the present paper, we will establish a necessary and sufficient optimality condition and some necessary optimality conditions for cone-preinvex set-valued functions in the topological vector space.

1 Notion and Preliminary Results

Let X and Y be topological vector spaces. A set-valued function F from X into Y is a function that associates a unique subset of Y with each point of X. Equivalently, F can be viewed as a function from X into the power set of Y, *i.e.* $F: X \to 2^Y$.

The domain of $F: X \to 2^Y$ is given by

$$D(F) = \{x(X \mid F(x) \neq \varnothing\}$$

For $E \subseteq X$, $F: E \to 2^Y$, denote, $F(E) = \bigcup_{x \in E} F(x)$.

A subset Γ of Y is said to be a cone if $\lambda\xi \in \Gamma$ for every $\xi \in \Gamma$, and $\lambda > 0$. A convex cone is one for which $\lambda_1\xi_1 + \lambda_2\xi_2 \in \Gamma$ for each $\xi_1, \xi_2 \in \Gamma$ and $\lambda_1, \lambda_2 \geqslant 0$. A pointed cone is one for which $\Gamma \cap (-\Gamma) = \{0\}$, where 0 is the zero element of Y. Let Γ be a pointed convex cone with int $\Gamma \neq \varnothing$.

Then we define three cone orders with respect to Γ as

$$\xi_1 \leqslant_\Gamma \xi_2 \quad \text{iff} \quad \xi_2 - \xi_1 \in \Gamma,$$
$$\xi_1 \leq_\Gamma \xi_2 \quad \text{iff} \quad \xi_2 - \xi_1 \in \Gamma \setminus \{0\},$$
$$\xi_1 < \xi_2 \quad \text{iff} \quad \xi_2 - \xi_1 \in \text{int}\, \Gamma.$$

The set of all the weak Γ-minimal points and weak Γ-maximal points of a set A in Y are defined

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as:

$$W - \mathrm{Min}_{\Gamma} A = \{y_0 \in A \mid \text{there exits no } y \in A \text{ for which } y <_{\Gamma} y_0\},$$

$$W - \mathrm{Max}_{\Gamma} A = \{y_0 \in A \mid \text{there exits no } y \in A \text{ for which } y_0 <_{\Gamma} y\}.$$

If $y_0 \in A$ is a weak minima of A with respect to cone Γ. then it is denoted by $y_0 \in W - \mathrm{Min}_{\Gamma} A$.

The polar cone Γ^* of Γ is defined as:

$$\Gamma^* = \{y^* \in Y^* \mid \langle y, y^* \rangle \geqslant 0 \text{ for all } y \in \Gamma\}.$$

The following result is due to Wang and Li[6].

Lemma 1.1 If $\Gamma \in Y$ is a pointed convex cone with int $\Gamma \neq \varnothing$, then

(1) $\Gamma + \mathrm{int}\,\Gamma \subset \mathrm{int}\,\Gamma$,

(2) $\langle y, y^* \rangle > 0$ for any $y^* \in \Gamma^* \setminus \{0\}$ and $y \in \mathrm{int}\,\Gamma$.

Definition 1.1[5] Let $E \subset X$ be a convex set and $F: E \to 2^Y$ be a set-valued function and Γ be a pointed convex cone in Y. Then F is said to be Γ—convex on E if for every $x_1, x_2 \in E, t \in [0,1]$.

$$tF(x_1) + (1-t)F(x_2) \subset F(tx_1 + (1-t)x_2) + \Gamma.$$

We define a new class of set-valued functions, called a preinvex set-valued function.

Definition 1.2[5] Let E be a subset of $X, F: E \to 2^Y$ and let Γ be a pointed convex cone in Y. F is said to be Γ—preinvex on E if there exits a function η defined on $X \times X$ and values in X such that for any $x_1, x_2 \in E, t \in [0,1]$.

$$tF(x_1) + (1-t)F(x_2) \subset F[x_2 + t\eta(x_1, x_2)] + \Gamma.$$

It is implicit in the above definition that for $x_1, x_2 \in E$, and $t \in [0,1], x_2 + t\eta(x_1, x_2) \in E$, we call such a set E to be an invex set with respect to η.

This definition generalizes the class of set-valued functions, as in the case where F is a Γ—convex function on E; then by taking $x_1 - x_2 = \eta(x_1, x_2)$ for all $x_1, x_2 \in E, F$ becomes Γ—preinvex. However, the converse need not be true ,that is, a Γ—preinvex set-valued function need not be Γ—convex.

The following theorem characterizes the generalized Farkas—Minkowski type theorem for preinvex set—valued functions.

Theorem 1.1[6] Let E be an invex subset of X(with respect to a function $\eta: X \times X \to X$). If the set-valued function $F: E \to 2^Y$ is Γ—preinvex and $G: E \to 2^Z$ is Λ—preinvex(with respect to some function η), where Γ and Λ are pointed convex cones in topdogical vector spaces Y and Z, respectively, then exactly one of the following statements is true:

(1)there exists $x \in E$ such that

$$F(x) \cap (-\mathrm{int}\,\Gamma) \neq \varnothing$$

$$G(x) \cap (-\mathrm{int}\,\Lambda) \neq \varnothing$$

(2)there exists $(y^*, z^*) \neq (0,0)$ in $\Gamma \times \Lambda$ such that for every $x \in E$,

$$\langle y^*, F(x) \rangle + \langle z^*, G(x) \rangle \geqslant 0.$$

The proof is given in [5].

Corollary 1.2 If in Theorem 1.1, we assume further that there exists $x^* \in E$ such that $G(x^*) \cap (-\mathrm{int}\,\Lambda) \neq \varnothing$, then $y^* \neq 0$.

Let Y, Z be ordered topological vector spaces with pointed convex cones Γ and Λ, respectively, the topological interiors of which are both nonempty. Then the product space $Y \times Z$ is also an ordered topological vector space with a pointed convex cones $\Gamma \times \Lambda$. We shall introduce below two common lemmas for the topological interior and the polar cone of $\Gamma \times \Lambda$.

Lemma 1.2 $\mathrm{int}(\Gamma \times \Lambda) = \mathrm{int}\,\Gamma \times \mathrm{int}\,\Lambda$.

Lemma 1.3 $(\Gamma\times\Lambda)^*=\Gamma^*\times\Lambda^*$.

The proofs of the two above Lemmas are easy.

2 Optimality Conditions

Let X be a topological vector space, and A,D be an invex subset of X (with respect to a function $\eta:X\times X\to X$). Let Y,Z be ordered topological vector spaces with pointed convex cones Γ and Λ, respectively, the topological interiors of which are both nonempty. Let $F:X\to 2^Y, G:X\to 2^Z$ be set-valued functions from X to Y and Z, respectively.

In this paper, we consider the following two classes of the optimization problems of set-valued functions

$$\min_{x\in A}F(x) \qquad (P1)$$

and

$$\min_{x\in D}F(x)$$
$$s.t.\ G(x)\cap(-\Lambda)\neq\varnothing \qquad (P2)$$

The feasible set of problem $(P2)$ is defined by

$$K=\{x(D|G(x)\cap(-\Lambda)\neq\varnothing\}.$$

Remark 1 Clearly, $y_0\in W-Min_\Gamma A$ iff $(A-y_0)\cap(-\text{int}\,\Gamma)=\varnothing$, where $A-y_0=\{y-y_0|y\in A\}$.

Definition 2.1 A point $x_0\in A$ is said to be a weak efficient solution of $(P1)$ if $\exists\ y_0\in F(x_0)$ such that $y_0\in W-\text{Min}_\Gamma F(A)$.

Definition 2.2 A point $x_0\in K$ is said to be a weak efficient solution of $(P2)$ if $\exists\ y_0\in F(x_0)$ such that $y_0\in W-\text{Min}_\Gamma F(K)$.

Clearly, $x_0\in A$ is a weak efficient solution of $(P1)$ iff $\exists\ y_0\in F(x_0)$ such that

$$[F(A)-y_0]\cap(-\text{int}\,\Gamma)=\varnothing.$$

and $x_0\in K$ is a weak efficient solution of $(P2)$ iff $\exists\ y_0\in F(x_0)$ such that

$$[F(K)-y_0]\cap(-\text{int}\,\Gamma)=\varnothing.$$

First, we consider the optimality condition for problem $(P1)$.

Theorem 2.1 Suppose that $F(x)$ is Γ-preinvex on A, and that $x_0\in A$. Then x_0 is a weak efficient solution of $(P1)$ iff there exists $y_0\in F(x_0)$, and $y^*\in\Gamma^*$, with $y^*\neq0$ such that

$$\inf\langle F(A),y^*\rangle=\langle y_0,y^*\rangle.$$

Proof. Necessity. By Definition 2.1, there exists $y_0\in F(x_0)$ such that $y_0\in W-\text{Min}_\Gamma F(A)$, *i.e.* $[F(x)-y_0]\cap(-\text{int}\,\Gamma)=\varnothing$, for all $x\in A$. It is clear that $F(x)-y_0$ is also Γ-preinvex on A, for $F(x)$ is Γ-preinvex on A. Thus, using Theorem 2.1, there exists $y^*\in\Gamma^*$, with $y^*\neq0$ such that

$$\langle F(A)-y_0,y^*\rangle\geqslant0,\ i.e.\ \langle F(A),y^*\rangle\geqslant\langle y_0,y^*\rangle.$$

However, $y_0\in F(x_0)$, therefore, $\inf\langle F(A),y^*\rangle=\langle y_0,y^*\rangle$.

Sufficiency. It follows directly from Theorem 1.1.

Q.E.D.

Now, we establish the optimality of $(P2)$. Let

$$H(x)=F(x)\times G(x), x\in X.$$

Then H is a set-valued function from X to product space $Y\times Z$ which is an ordered topological vector space with pointed convex cone $\Gamma\times\Lambda$ with a nonempty topological interior.

Theorem 2.2 Suppose the following:

1) $x_0\in K$ is a weak efficient solution of $(P2)$,

(2) $G(x)$ is Λ-preinvex on D and $H(x)$ is $\Gamma\times\Lambda$-preinvex on K.

Then there exists $y_0\in F(x_0)$, and $y^*\in\Gamma^*$, $z^*\in\Lambda^*$, with $(y^*,z^*)\neq(0,0)$ such that

$$\inf[\langle F(x),y^*\rangle+\langle G(x),z^*\rangle]=\langle y_0,y^*\rangle,$$
$$\inf<G(x_0),z^*\rangle=0.$$

Proof. According to Definition 2. 2, $\exists$ $y_0\in F(x_0)$ such that

$$[F(K)-y_0]\cap(-int\,\Gamma)=\varnothing. \tag{1}$$

For any $x\in X$, we have $[F(x)-y_0]\times G(x)=F(x)\times G(x)-(y_0,0)$. Let $H^*(x)=H(x)-(y_0,0)$, Since H is $\Gamma\times\Lambda$-preinvex on K, of course, H^* is also $\Gamma\times\Lambda$-preinvex on K. We have that

$$H^*(x)\cap[-int(\Gamma\times\Lambda)]=\varnothing,\text{for all } x\in K. \tag{2}$$

Suppose not. Then $\exists$ $x'\in K$ such that $H^*(x')\cap[-int(\Gamma\times\Lambda)]\neq\varnothing$. Hence, it follows by Lamma1. 2 that $[F(x')-y_0]\cap(-int\,\Gamma)\neq\varnothing$. Which contradicts (1). Therefore (2) holds. Thus, by Theorem 1. 1 and Lemma 1. 3, $\exists$ $y^*\in\Gamma^*$, $z^*\in\Lambda^*$, with $(y^*,z^*)\neq(0,0)$ such that

$$\langle H^*(x),(y^*,z^*)\rangle\geqslant 0,\text{ for any } x\in K,$$

It follows that

$$\langle F(x),y^*\rangle+\langle G(x),z^*\rangle\geqslant\langle y_0,y^*\rangle,\text{ for any } x\in K, \tag{3}$$

Due to $x_0\in K$, consequently $\exists$ $p\in G(x_0)$ such that $p\in(-\Lambda)$. But $z^*\in\Lambda^*$, which implies that $\langle p,z^*\rangle\leqslant 0$. On the other hand, to take $x=x_0$ in (3), we may get $\langle y_0,y^*\rangle+\langle p,z^*\rangle\geqslant\langle y_0,y^*\rangle$.
It follows that $\langle p,z^*>\geqslant 0$. So $\langle p,z^*\rangle=0$. Thus, we have $<y_0,y^*>\in\langle F(x_0),y^*\rangle+\langle G(x_0),z^*>$.

Hence, it follows from (3) that $\inf[\langle F(x),y^*\rangle+\langle G(x),z^*\rangle]=<y_0,y^*>$.

Take again $x=x_0$ in (3), we may get

$$\langle y_0,y^*\rangle+\langle G(x_0),z^*\rangle\geqslant\langle y_0,y^*\rangle.$$

So $\langle G(x_0),z^*\rangle\geqslant 0$, we have previously shown that there exists $p\in G(x_0)$ such that $\langle p,z^*\rangle=0$. Thus, $\inf\langle G(x_0),z^*\rangle=0$.

Q. E. D.

Corollary 2. 1 Suppose the following:

1) $x_0\in K$ is a weak efficient solution of $(P2)$,

2) D be an invex subset of X (with respect to a function $\eta:X\times X\to X$). F and G are Γ-preinvex and Λ-preinvex on D, respectively. Then $\exists$ $y_0\in(F(x_0)$, and $y^*\in\Gamma^*$, $z^*\in\Lambda^*$, with $(y^*,z^*)\neq(0,0)$ such that

$$\inf[\langle F(x),y^*\rangle+\langle G(x),z^*\rangle]=\langle y_0,y^*\rangle,$$
$$\inf\langle G(x_0),z^*\rangle=0.$$

Proof. Let $x_1,x_2\in D$, $\lambda\in[0,1]$. According to assumption 2), with respect to the same function $\eta:X\times X\to X$, we have

$$F(x_1)+(1-\lambda)F(x_2)\subset F(x_2+\lambda\eta(x_1,x_2))+\Gamma,$$
$$G(x_1)+(1-\lambda)G(x_2)\subset G(x_2+\lambda\eta(x_1,x_2))+\Lambda. \tag{4}$$

Clearly,

$$\lambda[F(x_1)\times G(x_1)]+(1-\lambda)[F(x_2)\times G(x_2)]=[F(x_1)+(1-\lambda)F(x_2)]\times[G(x_1)+(1-\lambda)G(x_2)].$$

Thus, by (4), we get

$$\lambda H(x_1)+(1-\lambda)H(x_2)\subset[F(x_2+\lambda\eta(x_1,x_2))+\Gamma]\times[G(x_2+\lambda\eta(x_1,x_2))+\Lambda]. \tag{5}$$

But the right-hand member of (5) is same as the set $F(x_2+\lambda\eta(x_1,x_2))\times G(x_2+\lambda\eta(x_1,x_2))+\Gamma$

$\times\Lambda$. Hence it follows from (5) that $\lambda H(x_1)+(1-\lambda)H(x_2)\subset H(x_2+\lambda\eta(x_1,x_2))+\Gamma\times\Lambda$, *i. e.* $H(x)$ is $\Gamma\times\Lambda$—preinvex on D. Now it is clear that feasible set K is invex, it follows that H is $\Gamma\times\Lambda$—preinvex on K.

Q. E. D.

We can similarly show the following theorem.

Theorem 2.3 Suppose the following:

1) $x_0\in K$ is a weak efficient solution of (P2),

2) $H(x)$ is $\Gamma\times\Lambda$—preinvex on D,

3) $[F(D\backslash K)-y_0]\cap(-int\ \Gamma)=\varnothing$, where y_0 is as in Theorem 2.1.

Then $\exists\ y^*\in\Gamma^*, z^*\in\Lambda^*$, with $(y^*,z^*)\neq(0,0)$ such that

$$\inf_{x\in D}[\langle F(x),y^*\rangle+\langle G(x),z^*\rangle]=\langle y_0,y^*\rangle,$$

$$\inf\langle G(x_0),z^*\rangle=0.$$

Theorem 2.4 Suppose the following:

1) $x_0\in K$,

2) $y_0\in F(x_0)$, and $(y^*,z^*)\in\Gamma^*\times\Lambda^*$, with $(y^*,z^*)\neq(0,0)$ such that

$$\inf[\langle F(x),y^*\rangle+\langle G(x),z^*\rangle]\geqslant\langle y_0,y^*\rangle,$$

3) $x'\in D$ such that $G(x')\cap(-int\ \Lambda)=\varnothing$.

Then x_0 is a weak efficient solution of $(P2)$.

Proof. By assumption 2), we have $\quad\langle F(x)-y_0,y^*\rangle+\langle G(x),z^*\rangle\geqslant0,\ \forall\ x\in D.\quad(6)$

First, we prove that $y^*\neq0$. Suppose not, *i. e.* $y^*=0$. Hence it follows from (6) that

$$\langle G(x),z^*\rangle\geqslant0,\ \forall\ x\in D.\tag{7}$$

By assumption 3), there exists $u\in G(x')$ such that $-u\in int\ \Lambda$, let $z\in Z$, then $\exists\ \lambda_0>0$ such that $-u+\lambda_0 z\in\Lambda$ and $-u-\lambda_0 z\in\Lambda$. since $z^*\in\Lambda^*$, thus

$$\langle-u+\lambda_0 z,z^*\rangle\geqslant0\text{ and }\langle-u-\lambda_0 z,z^*\rangle\geqslant0.\tag{8}$$

From (7), we can get that $\langle u,z^*\rangle\geqslant0$, hence it follows from (8) that $\langle z,z^*\rangle=0$, this implies that $z^*=0$, in contradiction to assumption (2), so $y^*\neq0$. If x_0 is not a weak efficient solution of $(P2)$, then $\exists\ x^*\in K$ such that $[F(x^*)-y_0]\cap(-int\ \Gamma)\neq\varnothing$, hence $\exists\ t\in F(x^*)$ such that $t-y_0\in(-int\ \Gamma)$. Since $y^*\in\Gamma^*$ and $y^*\neq0$, using Lemma 1.1, we obtain

$$\langle t-y_0,y^*\rangle<0\tag{9}$$

Due to $x^*\in K$, this implies that there exists $q\in G(x^*)$ such that $q\in(-\Lambda)$, it follows that

$$\langle q,z^*\rangle\leqslant0\tag{10}$$

Adding (10) to (9), we get $\langle t-y_0,y^*\rangle+\langle q,z^*\rangle<0$, which contradicts (6). thus x_0 is a weak efficient solution of $(P2)$.

Q. E. D

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锥一准不变凸集值映射的最优性条件

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摘要：研究了拓扑向量空间中的锥一准不变凸集值映射的极小值问题，得到了锥一准不变凸集值映射的最优性充要条件。

关 键 词：锥一准不变凸集值映射；最优性条件；弱有效解