四边固支矩形厚板分析的有限积分变换法

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摘 要:利用二维有限积分变换的方法推导出了四边固支矩形厚板位移和内力的精确解。弹性矩形厚板控制方程采用Mindlin三变量理论,在求解过程中不需要预先人为选取位移函数,而是直接对控制方程进行二维有限积分变换,将偏微分方程组化为简单的线性方程组进行求解,然后进行相应的积分逆变换得到实际问题的精确解。仅使用有限积分变换的数学方法,推导出了完全满足四边固支边界条件的矩形厚板问题的位移与内力的表达式,并对实例进行了数值计算。计算结果表明,运用有限积分变换的方法计算出的四边固支矩形厚板问题的位移和内力是精确的。

关键词:积分方程;厚板弯曲;有限积分变换;精确解;四边固支
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Analytical Solution of Clamped Rectangular Thick Plate by Finite Integral Transform Method

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Abstract: The exact solutions of displacement and inner force for clamped rectangular thick plate were deduced with two-dimension finite integral transform method. With the employment of Mindlin theory for solving the control equation of elastic rectangular thick plate, two-dimension finite integral transform was carried out for control equation without selecting displacement equation. And the high order partial differential equations were transformed to linear equations, and then the exact solutions can be obtained with integral inverted transform. The expression of displacement and inner force for clamped rectangular thick plate were deduced only with finite integral transform method and a case study was made for validation. And it was shown that the results from finite integral transform method for displacement and inner force are accurate.

Keywords: integral equations; thick plate bending; finite integral transform; exact solution; four edges clamped support

弹性矩形厚板是土木工程中较常见的一种结构形式,例如重型飞机跑道、集装箱运输车辆场坪、航天发射场坪、高层建筑筏板基础、船坞底板等。但是,由于厚板基本方程比薄板基本方程复杂得多以及数学方面的原因使得求其解析解是非常困难的。许多学者在求弹性矩形厚板的弯曲问题时提出了各种方法,主要可分为数值法和解析法两大类。数值法主要为有限元法[1-4]、边界元法[5-7],但这类方法的缺点是输入输出量大,计算较麻烦。解析法有三角级数法和叠加法[8],这两种方法可以得到解析解,但求解过程中不但非常复杂,并且需要事先人为的选择好位移函数,而位移函数的选取具有一定的任意性,无规律可寻。而近年来将辛几何引入弹性力学问题的求解成为一个研究热点[9-12]。

积分变换法是求解微分方程以及弹性力学问题解析解的较好的方法之一[13],该文从弹性厚板问题

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的基本方程出发，采用二维有限积分变换方法求出了完全满足四边固支边界条件矩形厚板位移和内力的精确解。由于求解过程中不需要人为选取位移函数，因此求解过程更加合理。

1 矩形厚板方程和有限积分变换

弹性矩形厚板的控制方程为[14-15]

$$\frac{\partial^2\varphi_x}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2\varphi_x}{\partial y^2}+\frac{1+\mu}{2}\frac{\partial^2\varphi_y}{\partial x\partial y}+\frac{C}{D}\left(\frac{\partial W}{\partial x}-\varphi_x\right)=0 \tag{1}$$

$$\frac{\partial^2\varphi_y}{\partial y^2}+\frac{1-\mu}{2}\frac{\partial^2\varphi_y}{\partial x^2}+\frac{1+\mu}{2}\frac{\partial^2\varphi_x}{\partial x\partial y}+\frac{C}{D}\left(\frac{\partial W}{\partial y}-\varphi_y\right)=0 \tag{2}$$

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}-\frac{\partial\varphi_x}{\partial x}-\frac{\partial\varphi_y}{\partial y}+\frac{q}{C}=0 \tag{3}$$

式中 $D=\frac{Eh^3}{12(1-\mu^2)}$ 为板的抗弯刚度；$C=\frac{5Eh}{12(1+\mu)}$ 为板的抗剪切刚度；E、h、μ 分别为板的弹性模量、厚度、泊松比；q 为作用于板表面的外力荷载。上述方程组含有三个广义位移，其中 $W(x,y)$ 为板的挠度函数，$\varphi_x(x,y)$、$\varphi_y(x,y)$ 分别是变形前垂直中面的直线段在 xz、yz 平面内的转角。由这三个广义位移可以求出内力，如板的弯矩为

$$M_x=-D\left(\frac{\partial\varphi_x}{\partial x}+\mu\frac{\partial\varphi_y}{\partial y}\right)$$

$$M_y=-D\left(\frac{\partial\varphi_y}{\partial y}+\mu\frac{\partial\varphi_x}{\partial x}\right) \tag{4}$$

四边固定矩形厚板的坐标如图1所示，根据文献[13]，对于任意函数 $f(x,y)$ 的二维有限正弦及余弦混合积分变换可以表示为

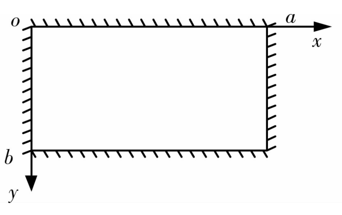

图1 弹性矩形厚板坐标示意图

$$\overline{F}^{ss}_{mn}=\int_0^a\int_0^b f(x,y)\sin\alpha_m x\sin\beta_n y\,\mathrm{d}x\mathrm{d}y \tag{5}$$

$$\overline{F}^{sc}_{mn}=\int_0^a\int_0^b f(x,y)\sin\alpha_m x\cos\beta_n y\,\mathrm{d}x\mathrm{d}y \tag{6}$$

$$\overline{F}^{cs}_{mn}=\int_0^a\int_0^b f(x,y)\cos\alpha_m x\sin\beta_n y\,\mathrm{d}x\mathrm{d}y \tag{7}$$

与其对应的逆变换分别为

$$f(x,y)=\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{F}^{ss}_{mn}\sin\alpha_m x\sin\beta_n y \tag{8}$$

$$f(x,y)=\frac{2}{ab}\sum_{m=1}^{\infty}\overline{F}^{sc}_{m0}\sin\alpha_m x+\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{F}^{sc}_{mn}\sin\alpha_m x\cos\beta_n y \tag{9}$$

$$f(x,y)=\frac{2}{ab}\sum_{n=1}^{\infty}\overline{F}^{cs}_{0n}\sin\beta_n y+\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{F}^{cs}_{mn}\cos\alpha_m x\sin\beta_n y \tag{10}$$

式中 $\alpha_m=\frac{m\pi}{a}$，$\beta_n=\frac{n\pi}{b}$，a、b 分别为矩形厚板的长度、宽度。

式(3)中各阶偏导数的有限正弦积分变换为

$$\int_0^a\int_0^b\frac{\partial^2 W}{\partial x^2}\sin\alpha_m x\sin\beta_n y\,\mathrm{d}x\mathrm{d}y=-\alpha_m\int_0^b\left[(-1)^m W|_{x=a}-W|_{x=0}\right]\sin\beta_n y\,\mathrm{d}y-\alpha_m^2\overline{W}^{ss}_{mn} \tag{11}$$

$$\int_0^a\int_0^b\frac{\partial^2 W}{\partial y^2}\sin\alpha_m x\sin\beta_n y\,\mathrm{d}x\mathrm{d}y=-\beta_n\int_0^a\left[(-1)^n W|_{y=b}-W|_{y=0}\right]\sin\alpha_m x\,\mathrm{d}x-\beta_n^2\overline{W}^{ss}_{mn} \tag{12}$$

$$\int_0^a\int_0^b\frac{\partial\varphi_x}{\partial x}\sin\alpha_m x\sin\beta_n y\,\mathrm{d}x\mathrm{d}y=-\alpha_m\overline{\varphi}^{cs}_{x\,mn} \tag{13}$$

$$\int_0^a\int_0^b\frac{\partial\varphi_y}{\partial y}\sin\alpha_m x\sin\beta_n y\,\mathrm{d}x\mathrm{d}y=-\beta_n\overline{\varphi}^{sc}_{y\,mn} \tag{14}$$

式(2)中各阶偏导数的有限混合积分变换为

$$\int_0^a\int_0^b\frac{\partial^2\varphi_y}{\partial x^2}\sin\alpha_m x\cos\beta_n y\,\mathrm{d}x\mathrm{d}y=-\alpha_m\int_0^b\left[(-1)^m\varphi_y|_{x=a}-\varphi_y|_{x=0}\right]\cos\beta_n y\,\mathrm{d}y-\alpha_m^2\overline{\varphi}^{sc}_{y\,mn} \tag{15}$$

$$\int_0^a\int_0^b\frac{\partial^2\varphi_y}{\partial y^2}\sin\alpha_m x\cos\beta_n y\,\mathrm{d}x\mathrm{d}y=\int_0^a\left[(-1)^n\frac{\partial\varphi_y}{\partial y}\Big|_{y=b}-\frac{\partial\varphi_y}{\partial y}\Big|_{y=0}\right]\sin\alpha_m x\,\mathrm{d}x-\beta_n^2\overline{\varphi}^{sc}_{y\,mn} \tag{16}$$

$$\int_0^a\int_0^b\frac{\partial^2\varphi_x}{\partial x\partial y}\sin\alpha_m x\cos\beta_n y\,\mathrm{d}x\mathrm{d}y=-\alpha_m\int_0^a\left[(-1)^n\varphi_x|_{y=b}-\varphi_x|_{y=0}\right]\cos\alpha_m x\,\mathrm{d}x-\alpha_m\beta_n\overline{\varphi}^{cs}_{x\,mn} \tag{17}$$

$$\int_0^a\int_0^b\frac{\partial W}{\partial y}\sin\alpha_m x\cos\beta_n y\,\mathrm{d}x\mathrm{d}y=\int_0^a\left[(-1)^n W|_{y=b}-W|_{y=0}\right]\sin\alpha_m x\,\mathrm{d}x+\beta_n\overline{W}^{ss}_{mn} \tag{18}$$

式(1)中各阶偏导数的有限混合积分变换为

$$\int_0^a\int_0^b\frac{\partial^2\varphi_x}{\partial x^2}\cos\alpha_m x\sin\beta_n y\,\mathrm{d}x\mathrm{d}y=\int_0^b\left[(-1)^m\frac{\partial\varphi_x}{\partial x}\Big|_{x=a}-\frac{\partial\varphi_x}{\partial x}\Big|_{x=0}\right]\sin\beta_n y\,\mathrm{d}y-\alpha_m^2\overline{\varphi}^{cs}_{x\,mn} \tag{19}$$

$$\int_0^a\int_0^b \frac{\partial^2 \varphi_x}{\partial y^2}\cos \alpha_m x \sin \beta_n y \mathrm{d}x\mathrm{d}y =$$
$$-\beta_n\int_0^a [(-1)^n\varphi_x|_{y=b}-\varphi_x|_{y=0}]\cos \alpha_m x \mathrm{d}x-\beta_n{}^2\,\overline{\varphi}_{x\,mn}^{cs} \tag{20}$$

$$\int_0^a\int_0^b \frac{\partial^2 \varphi_y}{\partial x \partial y}\cos \alpha_m x \sin \beta_n y \mathrm{d}x\mathrm{d}y =$$
$$-\beta_n\int_0^b [(-1)^m\varphi_y|_{x=a}-\varphi_y|_{x=0}]\cos \beta_n y \mathrm{d}y-\alpha_m\beta_n\,\overline{\varphi}_{y\,mn}^{sc} \tag{21}$$

$$\int_0^a\int_0^b \frac{\partial W}{\partial x}\cos \alpha_m x \sin \beta_n y \mathrm{d}x\mathrm{d}y =$$
$$\int_0^b [(-1)^m W|_{x=a}-W|_{x=0}]\sin \beta_n y \mathrm{d}y+\alpha_m\,\overline{W}_{mn}^{ss} \tag{22}$$

对外力进行有限正弦积分变换后得

$$\overline{q}_{mn}^{ss}=\int_0^a\int_0^b q(x,y)\sin \alpha_m x \sin \beta_n \mathrm{d}x\mathrm{d}y$$
$$q(x,y)=\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{q}_{mn}^{ss}\sin \alpha_m x \sin \beta_n y \tag{23}$$

四边固支厚板边界条件为

$$x=0,a:W=0,\varphi_y=0,\varphi_x=0 \tag{24}$$
$$y=0,b:W=0,\varphi_x=0,\varphi_y=0 \tag{25}$$

式(1)、(2)进行有限混合积分变换,式(3)进行有限正弦积分变换,将式(11)－(23)及边界条件式(24)、(25)前两项代入,得

$$\left(\frac{C\alpha_m}{D}\right)\overline{W}_{mn}^{ss}-\left[\alpha_m^2+\frac{\beta_n^2(1-\mu)}{2}+\frac{C}{D}\right]\overline{\varphi}_{x\,mn}^{cs}-\left[\frac{\alpha_m\beta_n(1+\mu)}{2}\right]\overline{\varphi}_{y\,mn}^{sc}=-(-1)^m K_n+L_n \tag{26}$$

$$\left(\frac{C\beta_n}{D}\right)\overline{W}_{mn}^{ss}-\left[\frac{\alpha_m\beta_n(1+\mu)}{2}\right]\overline{\varphi}_{x\,mn}^{cs}-\left[\beta_n^2+\frac{\alpha_m^2(1-\mu)}{2}+\frac{C}{D}\right]\overline{\varphi}_{y\,mn}^{sc}=-(-1)^n I_m+J_m \tag{27}$$

$$-(\alpha_m^2+\beta_n^2)\overline{W}_{mn}^{ss}+\alpha_m\,\overline{\varphi}_{x\,mn}^{cs}+\beta_n\,\overline{\varphi}_{y\,mn}^{sc}=\frac{-\overline{q}_{mn}^{ss}}{C} \tag{28}$$

其中令定积分常量

$$I_m=\int_0^a\frac{\partial\varphi_y}{\partial y}\Big|_{y=b}\sin \alpha_m x\mathrm{d}x,\ J_m=\int_0^a\frac{\partial\varphi_y}{\partial y}\Big|_{y=0}\sin \alpha_m x\mathrm{d}x$$
$$K_n=\int_0^b\frac{\partial\varphi_x}{\partial x}\Big|_{x=a}\sin \beta_n y\mathrm{d}y,\ L_n=\int_0^b\frac{\partial\varphi_x}{\partial x}\Big|_{x=0}\sin \beta_n y\mathrm{d}y \tag{29}$$

解式(26)-(28)联立的三元一次线性方程组,得

$$\overline{\varphi}_{x\,mn}^{cs}=\frac{\alpha_m\,\overline{q}_{mn}^{ss}}{D(\alpha_m^2+\beta_n^2)^2}+\frac{\alpha_m\beta_n}{\alpha_m^2+\beta_n^2}\left[\frac{1}{\alpha_m^2+\beta_n^2}-\frac{2D}{2C+D(1-\mu)(\alpha_m^2+\beta_n^2)}\right][(-1)^n I_m-J_m]+\left\{\frac{2D\beta_n^2}{(\alpha_m^2+\beta_n^2)[2C+D(1-\mu)(\alpha_m^2+\beta_n^2)]}+\frac{\alpha_m^2}{(\alpha_m^2+\beta_n^2)^2}\right\}[(-1)^m K_n-L_n] \tag{30}$$

$$\overline{\varphi}_{y\,mn}^{sc}=\frac{\beta_n\,\overline{q}_{mn}^{ss}}{D(\alpha_m^2+\beta_n^2)^2}+\frac{\alpha_m\beta_n}{\alpha_m^2+\beta_n^2}\left[\frac{1}{\alpha_m^2+\beta_n^2}-\frac{2D}{2C+D(1-\mu)(\alpha_m^2+\beta_n^2)}\right][(-1)^m K_n-L_n]+\left\{\frac{2D\alpha_m^2}{(\alpha_m^2+\beta_n^2)[2C+D(1-\mu)(\alpha_m^2+\beta_n^2)]}+\frac{\beta_n^2}{(\alpha_m^2+\beta_n^2)^2}\right\}[(-1)^n I_m-J_m] \tag{31}$$

$$\overline{W}_{mn}^{ss}=\frac{\overline{q}_{mn}^{ss}}{(\alpha_m^2+\beta_n^2)[D(\alpha_m^2+\beta_n^2)+C]}+\frac{\alpha_m[(-1)^m K_n-L_n]+\beta_n[(-1)^n I_m-J_m]}{(\alpha_m^2+\beta_n^2)^2} \tag{32}$$

由式(23)、(29)可知,$\overline{q}_{0n}^{ss}=\overline{q}_{m0}^{ss}=0$,$I_0=J_0=K_0=L_0=0$。则由式(30)、(31)得

$$\overline{\varphi}_{x\,0n}^{cs}=\frac{2D(K_n-L_n)}{[2C+D(1-\mu)\beta_n^2]}$$
$$\overline{\varphi}_{y\,m0}^{sc}=\frac{2D(I_m-J_m)}{[2C+D(1-\mu)\alpha_m^2]} \tag{33}$$

将式(30)－(33)代入式(8)－(10)定义的有限积分的逆变换,可以得到

$$W(x,y)=\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\overline{q}_{mn}{}^{ss}}{(\alpha_m{}^2+\beta_n{}^2)[D(\alpha_m{}^2+\beta_n{}^2)+C]}+\frac{\beta_n[(-1)^n I_m-J_m]+\alpha_m[(-1)^m K_n-L_n]}{(\alpha_m{}^2+\beta_n{}^2)^2}\right\}\sin \alpha_m x\sin \beta_n y \tag{34}$$

$$\varphi_x(x,y)=\frac{2}{ab}\sum_{n=1}^{\infty}\frac{2D(K_n-L_n)}{[2C+D(1-\mu)\beta_n^2]}\sin \beta_n y+\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\alpha_m\,\overline{q}_{mn}^{ss}}{D(\alpha_m^2+\beta_n^2)^2}+\frac{\alpha_m\beta_n}{\alpha_m^2+\beta_n^2}\left\{\frac{1}{\alpha_m^2+\beta_n^2}-\frac{2D}{[2C+D(1-\mu)(\alpha_m^2+\beta_n^2)]}\right\}[(-1)^n I_m-J_m]+\left\{\frac{\alpha_m^2}{\alpha_m^2+\beta_n^2}+\frac{2D\beta_n^2}{[2C+D(1-\mu)(\alpha_m^2+\beta_n^2)]}\right\}\times\frac{1}{\alpha_m^2+\beta_n^2}[(-1)^m K_n-L_n]\right\}\cos \alpha_m x\sin \beta_n y \tag{35}$$

$$\varphi_y(x,y)=\frac{2}{ab}\sum_{m=1}^{\infty}\frac{2D(I_m-J_m)}{[2C+D(1-\mu)\alpha_m^2]}\sin \alpha_m x+\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\beta_n\,\overline{q}_{mn}^{ss}}{D(\alpha_m^2+\beta_n^2)^2}+\frac{1}{\alpha_m^2+\beta_n^2}\left\{\frac{\beta_n^2}{\alpha_m^2+\beta_n^2}+\frac{2D\alpha_m^2}{[2C+D(1-\mu)(\alpha_m^2+\beta_n^2)]}\right\}[(-1)^n I_m-J_m]+\left\{\frac{1}{\alpha_m^2+\beta_n^2}-\frac{2D}{[2C+D(1-\mu)(\alpha_m^2+\beta_n^2)]}\right\}\times\frac{\alpha_m\beta_n}{\alpha_m^2+\beta_n^2}[(-1)^m K_n-L_n]\right\}\sin \alpha_m x\cos \beta_n y \tag{36}$$

为求得 I_m, J_m, K_n, L_n，将式(35)、(36)代入边界条件式(24)、(25)第3项，得

$$\frac{2}{ab}\sum_{n=1}^{\infty}\left\{\overline{\varphi}_{x0n}^{cs}+2\sum_{m=1}^{\infty}(-1)^m\overline{\varphi}_{xmn}^{cs}\right\}\sin\beta_n y=0 \tag{37}$$

$$\frac{2}{ab}\sum_{n=1}^{\infty}\left\{\overline{\varphi}_{x0n}^{cs}+2\sum_{m=1}^{\infty}\overline{\varphi}_{xmn}^{cs}\right\}\sin\beta_n y=0 \tag{38}$$

$$\frac{2}{ab}\sum_{m=1}^{\infty}\left\{\overline{\varphi}_{ym0}^{sc}+2\sum_{n=1}^{\infty}(-1)^n\overline{\varphi}_{ymn}^{sc}\right\}\sin\alpha_m x=0 \tag{39}$$

$$\frac{2}{ab}\sum_{m=1}^{\infty}\left\{\overline{\varphi}_{ym0}^{y}+2\sum_{n=1}^{\infty}\overline{\varphi}_{ymn}^{y}\right\}\sin\alpha_m x=0 \tag{40}$$

式(37)、(38)，式(39)、(40)分别相减、相加，则式(24)、(25)第3项表示的边界条件可化为

$$\sum_{m=1,3\cdots}^{\infty}\overline{\varphi}_{xmn}^{cs}=0,\overline{\varphi}_{x0n}^{cs}+2\sum_{m=2,4\cdots}^{\infty}\overline{\varphi}_{xmn}^{cs}=0,n=1,2\cdots\infty \tag{41}$$

$$\sum_{n=1,3\cdots}^{\infty}\overline{\varphi}_{ymn}^{sc}=0,\overline{\varphi}_{ym0}^{sc}+2\sum_{n=2,4\cdots}^{\infty}\overline{\varphi}_{ymn}^{sc}=0,m=1,2\cdots\infty \tag{42}$$

将式(30)－(33)代入式(41)、(42)并计算无穷级数极限和，分别得到

$$\sum_{m=1,3\cdots}^{\infty}\left[\frac{1}{\alpha_m^2+\beta_n^2}-\frac{2D}{2C+D(1-\mu)(\alpha_m^2+\beta_n^2)}\right]\times\frac{\alpha_m\beta_n}{\alpha_m^2+\beta_n^2}[(-1)^n I_m-J_m]-\left\{\frac{a^2}{8\pi\varepsilon_n}\left[\tanh\frac{\pi\varepsilon_n}{2}+\frac{\frac{\pi\varepsilon_n}{2}}{\cosh^2\left(\frac{\pi\varepsilon_n}{2}\right)}\right]+\frac{D\pi\varepsilon_n^2}{4C}\left[\frac{\tanh\left(\frac{\pi\varepsilon_n}{2}\right)}{\varepsilon_n}-\frac{\tanh\left(\frac{\pi\gamma_n}{2}\right)}{\gamma_n}\right]\right\}(K_n+L_n)=-\sum_{m=1,3\cdots}^{\infty}\frac{\alpha_m\overline{q}_{mn}^{ss}}{D(\alpha_m^2+\beta_n^2)^2},n=1,2\cdots\infty \tag{43}$$

$$\sum_{m=2,4\cdots}^{\infty}\left[\frac{1}{\alpha_m^2+\beta_n^2}-\frac{2D}{2C+D(1-\mu)(\alpha_m^2+\beta_n^2)}\right]\times\frac{\alpha_m\beta_n}{\alpha_m^2+\beta_n^2}[(-1)^n I_m-J_m]+\left\{\frac{D}{2C+D(1-\mu)\beta_n^2}+\left\{\frac{a^2}{8\pi\varepsilon_n}\left[2\left(\coth\pi\varepsilon_n-\frac{\pi\varepsilon_n}{\sinh^2\pi\varepsilon_n}\right)-\left(\tanh\frac{\pi\varepsilon_n}{2}+\frac{\frac{\pi\varepsilon_n}{2}}{\cosh^2\left(\frac{\pi\varepsilon_n}{2}\right)}\right)\right]+\frac{D\pi\varepsilon_n^2}{4C}\left[2\left(\frac{\coth\pi\varepsilon_n}{\varepsilon_n}-\frac{\coth\pi\gamma_n}{\gamma_n}\right)-\left(\frac{\tanh\left(\frac{\pi\varepsilon_n}{2}\right)}{\varepsilon_n}-\frac{\tanh\left(\frac{\pi\gamma_n}{2}\right)}{\gamma_n}\right)\right]-\xi_n\right\}\right\}(K_n-L_n)=-\sum_{m=2,4\cdots}^{\infty}\frac{\alpha_m\overline{q}_{mn}^{ss}}{D(\alpha_m^2+\beta_n^2)^2},n=1,2\cdots\infty \tag{44}$$

$$-\left\{\frac{b^2}{8\pi^2\varepsilon_m}\left[\tanh\frac{\pi\varepsilon_m}{2}+\frac{\frac{\pi\varepsilon_m}{2}}{\cosh^2\left(\frac{\pi\varepsilon_m}{2}\right)}\right]+\frac{D\pi\varepsilon_m^2}{4C}\times\left[\frac{\tanh\left(\frac{\pi\varepsilon_m}{2}\right)}{\varepsilon_m}-\frac{\tanh\left(\frac{\pi\gamma_m}{2}\right)}{\gamma_m}\right]\right\}(I_m+J_m)+\sum_{n=1,3\cdots}^{\infty}\left[\frac{1}{\alpha_m^2+\beta_n^2}-\frac{2D}{2C+D(1-\mu)(\alpha_m^2+\beta_n^2)}\right]\times\frac{\alpha_m\beta_n}{\alpha_m^2+\beta_n^2}[(-1)^m K_n-L_n]=-\sum_{n=1,3\cdots}^{\infty}\frac{\beta_n\overline{q}_{mn}^{ss}}{D(\alpha_m^2+\beta_n^2)^2},m=1,2\cdots\infty \tag{45}$$

$$\left\{\frac{D}{2C+D(1-\mu)\alpha_m^2}+\frac{b^2}{8\pi\varepsilon_m}\left[2\left(\coth\pi\varepsilon_m-\frac{\pi\varepsilon_m}{\sinh^2\pi\varepsilon_m}\right)-\left(\tanh\frac{\pi\varepsilon_m}{2}+\frac{\frac{\pi\varepsilon_m}{2}}{\cosh^2\left(\frac{\pi\varepsilon_m}{2}\right)}\right)\right]+\frac{D\pi\varepsilon_m^2}{4C}\left[2\left(\frac{\coth\pi\varepsilon_m}{\varepsilon_m}-\frac{\coth\pi\gamma_m}{\gamma_m}\right)-\left(\frac{\tanh\left(\frac{\pi\varepsilon_m}{2}\right)}{\varepsilon_m}-\frac{\tanh\left(\frac{\pi\gamma_m}{2}\right)}{\gamma_m}\right)\right]-\xi_m\right\}(I_m-J_m)+\sum_{n=2,4\cdots}^{\infty}\left[\frac{1}{\alpha_m^2+\beta_n^2}-\frac{2D}{2C+D(1-\mu)(\alpha_m^2+\beta_n^2)}\right]\times\frac{\alpha_m\beta_n}{\alpha_m^2+\beta_n^2}[(-1)^m K_n-L_n]=-\sum_{n=2,4\cdots}^{\infty}\frac{\beta_n\overline{q}_{mn}^{ss}}{D(\alpha_m^2+\beta_n^2)^2},m=1,2\cdots\infty \tag{46}$$

其中

$$\varepsilon_n=\frac{na}{b},\gamma_n=\sqrt{\varepsilon_n^2+\frac{2Ca^2}{D(1-\mu)\pi^2}},$$

$$\xi_n=\frac{Da^2}{[D(1-\mu)\pi^2\varepsilon_n^2+2Ca^2]},$$

$$\varepsilon_m=\frac{mb}{a},\gamma_m=\sqrt{\varepsilon_m^2+\frac{2Cb^2}{D(1-\mu)\pi^2}},$$

$$\xi_m=\frac{Db^2}{[D(1-\mu)\pi^2\varepsilon_m^2+2Cb^2]}。$$

利用 Matlab 由式(43)－(46)计算出 $I_m, J_m, K_n, L_n(m,n=1,2\cdots\infty)$，代入式(34)－(36)即可求得完全满足由式给出的边界条件的四边固支矩形厚板问题的精确解。

2 算例

为了验证该方法的正确性，以文献[15]中均布荷载作用于四边固支矩形厚板为例，取 $\frac{a}{b}=1,\mu=0.3$，$\delta=\frac{D}{Ca^2}$，计算厚板中点挠度与固支边中点的弯矩。表1、表2列出该方法计算结果与文献[15]结果对比。

表 1　均布荷载作用下厚板中点挠度

δ	$W\mid_{x=a/2,y=b/2}/qa^4\cdot D^{-1}$	
	文献[15]	该方法
0(薄板解)	0.001 26	0.001 265 3
0.005	0.001 68	0.001 679 6
0.010	0.002 07	0.002 066 7
0.015	0.002 45	0.002 446
0.020	0.002 82	0.002 82
0.025	0.003 20	0.003 200 2
0.030	0.003 57	0.003 571 5
0.040	0.004 32	0.004 328 6
0.050	0.005 06	0.005 054 4

表 2　均布荷载作用下固支边中点弯矩值

δ	$M_x\mid_{x=0,y=b/2}/(qa^2)$	
	文献[15]	该方法
0(薄板解)	−0.051 3	−0.051 333
0.005	−0.048 1	−0.048 133
0.010	−0.046 4	−0.046 371
0.015	−0.045 3	−0.045 277
0.020	−0.044 5	−0.044 531
0.025	−0.044 0	−0.043 988
0.030	−0.043 6	−0.043 576
0.040	−0.043 0	−0.042 99
0.050	−0.042 6	−0.042 594

以上计算结果表明，该方法的计算结果同文献[15]的计算结果是一致的，表明利用二维有限积分变换的方法所求得的四边固支矩形厚板的结果是正确的。

3　结　语

利用二维有限积分变换的数学方法求出了四边固支矩形厚板位移与内力的精确解。求解过程中不需要事先人为的选取位移函数，而是从弹性厚板问题的基本方程出发，直接利用数学的方法求出完全满足四边固支边界条件矩形厚板问题的精确解，通过数值算例证明了该方法的正确性。

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