Topological inverse semigroups

ZHU Yongwen
Department of Mathematics and Information Science, Yantai University, Yantai City 264005, P. R. China,
Received 20 February 2004; revised 26 May 2004

Abstract: That the projective limit of any projective system of compact inverse semigroups is also a compact inverse semigroup, the injective limit of any injective system of inverse semigroups is also an inverse semigroup, and that a compact inverse semigroup is topologically isomorphic to a strict projective limit of compact metric inverse semigroups are proved. It is also demonstrated that Hom(S,T) is a topological inverse semigroup provided that S or T is a topological inverse semigroup with some other conditions. Being proved by means of the combination of topological semigroup theory with inverse semigroup theory, all these results generalize the corresponding ones related to topological semigroups or topological groups.

Keywords: topological inverse semigroup; compact; projective limit; injective limit

1. Introduction

A topological semigroup is a triple (X,*,τ), where (X,τ) is a topological Hausdorff space and * is a continuous associative binary operation on X [1]. If, moreover, (X,*) is an inverse semigroup and the inverse operation is continuous, then it is said that (X,*,τ) is a topological inverse semigroup. If the word “semigroup” appears with a topological adjective, then “topological semigroup” is implied. For example, the statement “S is a compact semigroup” means that S is a compact topological semigroup [2]. Observe that any inverse semigroup can be made into a topological inverse semigroup by giving it the discrete topology, and thus a finite inverse semigroup is a compact inverse semigroup. The following example can be found in Ref.[3].

On the set (−∞,∞)×(−∞,∞), define a multiplication such that (a,b)(c,d) = (a+c−min{b,c},b+d−min{b,c}) and a topology such that a base of neighborhoods of (a,b) is

{(x,y): y−x = b−a & |a−x|< ε; ε > 0}.

The set (−∞,∞)×(−∞,∞) with the multiplication and topology defined so forms a locally compact topological inverse semigroup.

There have been sufficient studies in topological semigroups and in inverse semigroups, respectively [1-5]. But topological inverse semigroups need more research. In this paper, a series of fundamental results on topological semigroups are generalized to the topological inverse semigroups.

The theory of inverse semigroups can be referred to Ref.[4]. If S is a semigroup, then denote the set of all idempotents of S by E_S. Let a,b be chosen in a semigroup such that aba = a and bab = b, then b is called an inverse of a. A semigroup is said to be regular if each of its elements has an inverse. Further, a semigroup in which every element has a unique inverse is called an inverse semigroup. The unique inverse of the element a in an inverse semigroup is denoted by a^{-1}. The following lemma will be useful.

Lemma 1.1. Let f be a homomorphism from an inverse semigroup A into another inverse semigroup B. Then f(a^{-1}) = f(a)^{-1} for any a ∈ A.

Proof. Let b = a^{-1}. Then aba = a and bab = b. Since f is a homomorphism, there exist f(a)f(b)f(a) = f(a) and f(b)f(a)f(b) = f(b). So f(a^{-1}) = f(b) = f(a)^{-1}, as desired.

If {X_a}_{α∈D} is a collection of topological spaces indexed by a nonempty set D, then denote the Cartesian product with the product topology by \prod{X_a}_{α∈D} and the \alpha projection for each \alpha ∈ D by \pi_α : \prod{X_a}_{α∈D} → X_α.

Lemma 1.2. Let {X_a}_{α∈D} and {Y_a}_{α∈D} be two collections of topological spaces, \{f_a\}_{α∈D} be a map from X_a into Y_a for each \alpha ∈ D, and \prod{f_a}_{α∈D} be defined by \prod(f_a)(x)(α) = f_α(x(α)). Then, \prod{f_a}_{α∈D} is continuous if and only if f_α is continuous for each \alpha ∈ D.

Proof. Let f = \prod{f_a}_{α∈D}, S = \prod{X_a}_{α∈D} and T = \prod{Y_a}_{α∈D}. For every open subset B of T, there exists a family of open subsets \{B_i\}_{i∈I} of T such that B = \bigcup_{i∈I} B_i and for each i ∈ I and each \alpha ∈ D,
Therefore, for showing the sufficiency, it suffices to show that $f^{-1}(B)$ is open whenever $B = \prod C_{a_{\alpha}}$ and $C_a$ is an open subset of $Y_a$ for every $\alpha \in D$. For every $\alpha \in D$, $f_{a_{\alpha}}^{-1}(C_a)$ is an open subset of $X_a$ because $f_a$ is continuous. So $f^{-1}(B) = \prod (f_{a_{\alpha}}^{-1}(C_a))_{a_{\alpha}}$ is an open subset of $X$, and the sufficiency holds. It is also straightforward to show the necessity.

A projective system of [topological] semigroups is a triple $((D, \leq), \{S_\alpha\}_{a_{\alpha} \in D}, \{\phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \leq})$ where

- $D$ is a directed set;
- $\{S_\alpha\}_{a_{\alpha} \in D}$ is a family of “topological” semigroups indexed by $D$; and
- $\{\phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \leq}$ is a family of functions indexed by $\leq$ such that
  - $\phi^\alpha_x : S_x \to S_\alpha$ is a “continuous” homomorphism for each $(\alpha, x) \in D$;
  - $\phi^\alpha_x = 1_{S_\alpha}$ for each $\alpha \in D$; and
  - $\phi^\alpha_x \circ \phi^\beta_x = \phi^\beta_x$ for all $\alpha \leq \beta \leq \gamma$ in $D$.

When no confusion is probable, denote the projective system

$$((D, \leq), \{S_\alpha\}_{a_{\alpha} \in D}, \{\phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \leq})$$

by $\{S_\alpha, \phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \in D}$. Each $\phi^\alpha_x$ is called a bonding map and $\{S_\alpha, \phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \in D}$ is said to be strict if each bonding map is surjective. If $S = \{x \in \prod \{S_\alpha\}_{a_{\alpha} \in D} : \phi^\alpha_x(x(\beta)) = x(\alpha) \}$ for all $\alpha \leq \beta$, then $S$ is called the projective limit of $\{S_\alpha, \phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \in D}$. If $\{S_\alpha, \phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \in D}$ is a strict projective system, then $S$ is called the strict projective limit of $\{S_\alpha, \phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \in D}$. Namely $S = \lim_{\alpha_{\alpha} \leq} \{S_\alpha, \phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \in D}$ or $S = \lim_{\alpha_{\alpha} \leq} S_\alpha$ if no confusion occurs.

An injective system of “topological” semigroups is a triple $((D, \leq), \{S_\alpha\}_{a_{\alpha} \in D}, \{\phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \leq})$ where

- $(D, \leq)$ is a directed set;
- $\{S_\alpha\}_{a_{\alpha} \in D}$ is a family of [topological] semigroups indexed by $D$; and
- $\{\phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \leq}$ is a family of functions indexed by $\leq$ such that
  - $\phi^\alpha_x : S_\alpha \to S_x$ is a “continuous” homomorphism for each $(\alpha, x) \in D$;
  - $\phi^\alpha_x = 1_{S_x}$ for each $\alpha \in D$; and
  - $\phi^\alpha_x \circ \phi^\beta_x = \phi^\beta_x$ for all $\alpha \leq \beta \leq \gamma$ in $D$.

If no confusion is probable, denote the projective system $((D, \leq), \{S_\alpha\}_{a_{\alpha} \in D}, \{\phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \leq})$ by $\{S_\alpha, \phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \in D}$.

Let $\{S_\alpha, \phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \in D}$ be an injective system of topological semigroups and let $A = \bigcup \{\{x\} \times S_\alpha : \alpha \in D\}$. Then by Lemma 2.30 of Ref. [1], $R = \{(x, (\alpha, y)) \in A \times A : \phi^y_x(x) = \phi^y_y(y)\}$ for some $\gamma \geq \alpha, \beta \in D$ is an equivalence on $A$. Denote $A/R$ by $S$. Let $\pi : A \to S$ be the natural map and define a multiplication on $S$ by the following way: for $a, b, c \in S, c = ab$ if and only if there exist $(a, x), (b, y), (c, z)$ in $A$ such that $a = \pi(x), b = \pi(y), c = \pi(z), \gamma \geq \alpha, \beta$, and $\phi^y_x(a) = \phi^y_y(b) = c$. By Theorem 2.31 of Ref. [1], $S$ is a topological semigroup, called the injective limit of the injective system $\{S_\alpha, \phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \in D}$, i.e. $S = \lim S_\alpha \phi^\alpha_{a_{\alpha}}$ or, when no confusion likely occurs, $S = \lim S_\alpha$.

In view of Theorem 2.22 of Ref. [1], the projective limit of the projective system of compact semigroups is still a compact semigroup. In view of Theorem 2.29 in Ref. [1], any compact semigroup is topologically isomorphic to a strict projective limit of compact metric semigroup. According to Theorem 2.31 in Ref. [1], the injective limit of the injective system of a semigroups group is also a semigroup group. Hereinafter, it will be proved that all of these statements remain valid if “semigroup” and “group” are replaced by “inverse semigroup” throughout.

For topological semigroups $S$ and $T$ with $T$ Abelian, $\Hom(S, T)$ is used to denote the set of all continuous homomorphisms from $S$ into $T$ with the compact-open topology and pointwise multiplication $((fg)(x)) = (f(x))g(x)$ for all $x \in S$. According to Theorem 2.33 of Ref. [1], if $S$ is a locally compact semigroup and $T$ is an Abelian topological semigroup such that $\Hom(S, T) \neq \emptyset$, then $\Hom(S, T)$ is an Abelian topological semigroup. Moreover, according to Theorem 2.34 of Ref. [1], if $T$ is a topological group, then $\Hom(S, T)$ is a topological group. It will be shown that $\Hom(S, T)$ is a topological inverse semigroup supposed that $S$ is a topological reverse semigroup and $\Hom(S, T) \neq \emptyset$, or $T$ is a topological inverse semigroup.

2. Main results

2.1 Projective limits of projective systems and generalization of Theorem 2.22 of Ref. [1].

**Theorem 2.1.** Let $\{S_\alpha, \phi^\alpha_{a_{\alpha}}\}_{a_{\alpha} \in D}$ be a projective system of compact inverse semigroups. Then
Let \( S = \lim S_n \) be a compact inverse semigroup.

**Proof.** In order to seek the regularity of \( S \), let \( x \in S \).

For each \( \alpha \in D \), there exists \( y(\alpha) \in S_n \) such that 
\[
(\alpha(\alpha))y(\alpha)x(\alpha) = x(\alpha) \quad \text{and} \quad y(\alpha)x(\alpha)y(\alpha) = y(\alpha)
\]
by the regularity of \( S_n \). Thus \( y(\alpha) = x(\alpha)^{-1} \). For each 
\( (\alpha, \beta) \in \Sigma \), \( \phi_{\beta}^{\alpha}(x(\beta)) = x(\alpha) \). In view of \( \text{Lemma } 1.1 \), 
\( \phi_{\beta}^{\alpha}(x(\beta))^{-1} = x(\alpha)^{-1} \). It follows that 
\[
\phi_{\beta}^{\alpha}(y(\beta)) = y(\alpha)
\]
Therefore, \( y \in S \). It is clear that \( xy \alpha x \) and 
\( yxy = y \). So \( S \) is regular.

To show that \( S \) is an inverse semigroup, let \( x, y \in E_S \). Then for each \( \alpha \in D \), \( x(\alpha), y(\alpha) \in E_S \).
So \( x(y(\alpha)) = y(\alpha)x(\alpha) \) because \( S_n \) is an inverse semigroup. Thus \( xy = xy \).

From the first paragraph of this proof, it is obvious that for each \( \alpha \in D \), \( x^{-1}(\alpha) = (x(\alpha))^{-1} \). For each 
\( \alpha \in D \), the inversion of \( S_n \) is continuous by the hypothesis. So according to \( \text{Lemma } 1.2 \), the continuity of the inversion of \( S \) is demonstrated.

2.2 Generalization of Theorem 2.26 in Ref. [1]

**Theorem 2.2.** Let \( S \) be a compact totally disconnected inverse semigroup. Then \( S \) is topologically isomorphic to a strict projective limit of finite discrete inverse semigroups.

**Proof.** Let 
\( R = \{ R : R \) is an open and closed congruence on \( S \}\). According to Theorem 2.26 and its proof in Ref. [1], it suffices to show that for any \( R \in R, S/ R \) is an inverse semigroup. In view of \( \text{Lemma } II 1.10 \) in Ref. [4], \( S/ R \) is an inverse semigroup because \( S \) is an inverse semigroup.

A metric \( d \) on a semigroup \( S \) is said to be subinvariant if for each \( a, x, y \in S \), 
\[
d(ax, ay) \leq d(x, y), \quad \text{and} \quad d(xa, ya) \leq d(x, y).
\]
A topological semigroup \( S \) is said to be a metric semigroup if there exists a subinvariant metric \( d \) on \( S \) which determines the topology of \( S \). Theorem 2.29 of Ref. [1] appears first in Ref. [2]. By generalizing this theorem, the following result is obtained.

**Theorem 2.3.** Let \( S \) be a compact inverse semigroup. Then \( S \) is topologically isomorphic to a strict projective limit of compact metric inverse semigroups.

**Proof.** Using Theorem 2.29 in Ref. [1] and its proof, the argument is similar to that of Theorem 2.2.

2.3 Injective limits and generalization of Theorem 2.31 in Ref. [1].

**Theorem 2.4.** Let \( \{ S_n, \phi_{\beta}^{\alpha} \}_{\alpha \geq 0} \) be an injective system of inverse semigroups. Then \( S = \lim S_n \) is an inverse semigroup.

**Proof.** To seek the regularity of \( S \), let \( a \in S \) and \( a = \pi \) with \( x \in S_n \). Assuming that \( S_n \) is an inverse semigroup. Thus \( S_n \) is regular. So there exists some \( y \in S_n \) such that \( xy = x \) and \( yxy = y \). Let \( b = \pi(\alpha, y) \). Then \( aba = a \) and \( bab = b \) by the definition of the multiplication of \( S \). Therefore, \( S \) is a regular semigroup.

Let \( e \in E_S \) and \( e = \pi(\alpha, x) \) with \( x \in S_n \). Assume that \( x^2 = y \). From \( e^2 = e \), it is readily seen that \( \pi(\alpha, x) = \pi(\alpha, y) \). By the definitions of \( \pi \) and \( R \), there exists some \( \beta \) in \( D \) such that \( \alpha \leq \beta \) and \( \phi_{\beta}^{\alpha}(x) = \phi_{\beta}^{\alpha}(y) \). Let \( z = \phi_{\beta}^{\alpha}(x) \). Then \( z = \phi_{\beta}^{\alpha}(x) = \phi_{\beta}^{\alpha}(y) = \phi_{\beta}^{\alpha}(x^2) = \phi_{\beta}^{\alpha}(x) \phi_{\beta}^{\alpha}(x) = z^2 \). Thus \( z \in E_{S_n} \) and \( e = \pi(\alpha, x = \pi(\beta, z) \).

By the last paragraph, there exist \( \alpha, \beta \in D \), 
\( x \in E_{S_n} \) and \( y \in E_{S_n} \) such that \( e = \pi(\alpha, x) \) and 
\( f = \pi(\alpha, y) \). Let \( \gamma \) be in \( D \) such that \( \alpha, \beta \leq \gamma \). Let 
\( u = \phi_{\beta}^{\alpha}(x) \) and \( v = \phi_{\beta}^{\alpha}(y) \). Then \( u, v \in E_{S_n} \) since \( \phi_{\beta}^{\alpha} \) and \( \phi_{\beta}^{\alpha} \) are homomorphisms. In light of Theorem II 1.2 in Ref. [4], there is \( uv = vu \) since \( S_n \) is an inverse semigroup. Moreover, 
\[
e d = \pi(\alpha, x) \pi(\alpha, y) = \pi(\gamma, uv) = \pi(\gamma, vu) = \pi(\alpha, y) \pi(\alpha, x) = fe, \quad \text{as required.}
\]
By Theorem II 1.2 of Ref. [4] again, \( S \) is an inverse semigroup.

2.4 Semigroups of homomorphisms and generalization of Theorem 2.34 in Ref. [1]

**Theorem 2.5.** Let \( S \) be a locally compact semigroup and \( T \) an Abelian topological semigroup. If 
1) \( S \) is a topological inverse semigroup and 
\( \text{Hom}(S, T) \neq \emptyset \), or 
2) \( T \) is a topological inverse semigroup, 
then \( \text{Hom}(S, T) \) is an Abelian topological inverse semigroup.

**Proof.** Observe that if \( T \) is an inverse semigroup then \( \text{Hom}(S, T) \neq \emptyset \) since for any \( b \in T \), letting \( f(x) = b^{-1}b \) for all \( x \in S \) leads to \( f \in \text{Hom}(S, T) \). So, for completing the proof, in view of Theorem 2.33 of Ref. [1], it suffices to show that \( \text{Hom}(S, T) \) is an inverse semigroup and its inversion is continuous.
In case (1), let $\tau$ be the inversion of $S$ and denote $f \circ \tau$ by $\overline{f}$ for any fin $\text{Hom}(S,T)$. Then for each $x \in S$, $\overline{f}(x) = f \circ \tau(x) = f(x^{-1})$. By the definition of the multiplication of $\text{Hom}(S,T)$ and the assumption that $f$ is a homomorphism, $(f \overline{f} f)(x) = f(x)\overline{f}(x) = f(x)f(x^{-1})f(x) = f(x)xx^{-1}x = f(x)$. Thus $f \overline{f} f = f$. Similarly, $\overline{f} f \overline{f} = f$. So, $\text{Hom}(S,T)$ is regular. Because $\text{Hom}(S,T)$ is a commutative semigroup, it is an inverse semigroup by Theorem II 1.2 of Ref. [4]. It is clear that $\overline{f} = f \circ \tau$ is continuous because $f$ and $\tau$ are both continuous.

In case (2), let $\tau$ be the inversion of $T$ and denote $\tau \circ f$ by $\overline{f}$ for any $f \in \text{Hom}(S,T)$. Then for each $x \in S$, $\overline{f}(x) = \tau \circ f(x) = f(x)^{-1}$. It follows that $(f \overline{f} f)(x) = f(x)\overline{f}(x)f(x) = f(x)f(x)^{-1}f(x) = f(x)$. Thus $f \overline{f} f = f$. Similarly, $\overline{f} f \overline{f} = f$. Hence $\text{Hom}(S,T)$ is regular. Furthermore, $\text{Hom}(S,T)$ is an inverse semigroup. Evidently, $\overline{f} = f \circ \tau$ is continuous because $f$ and $\tau$ are both continuous.

References