Numerical simulation of crack propagation by manifold method *

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Abstract: In this paper, by means of the maximum circle tensile stress on curve of constant $\omega$ and stress intensity factors by a path independent contour integral method, trajectories of maxed mode crack propagation are simulated through numerical manifold method. The crack propagation is traced dynamically by modifying the neighboring connection between the crack-top and nodes within elements in the calculating process. This method has the advantages such as less modified area, easiness of programming, high realizability and so on. Then a single sharp nicked specimen is used to verified the numerical result. It is shown that the provided method is reasonable and effective.

Keywords: numerical manifold method; crack propagation; path independent contour integral

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1 Introduction

The crack problem has always been an important topic to researchers in engineering mechanism. Some initial defects, like microcrack for example, will exist in the manufacturing procedure of real materials, they will grow or coalesce gradually in the following utility process due to the effects such as loads, and then initiate the visual macrocracks finally. Mechanical analysis of damaged structural components should be required in not only designing new components but also predicting used components, to determine its ability against loads. The common situation in engineering is mixed-mode crack problem, which is difficult or impossible to obtain its analytic solution. Several efficient numerical techniques for crack problems, such as the finite elements method (FEM), have been developed in the past decade years. However, to apply FEM to studying the crack propagation behavior will greatly increase the complexity of program design and the numerical calculation because it requires continual remeshing after each crack increment at a variety of loads, which has limited to a great degree the extensive application of FEM to structural analysis.

The traditional FEM plays a master role in numerical simulation of engineering problem, due to its maturity in theory, simplicity in application, powerful support of commercial software. Therefore any complementarity and development of FEM will the have great effect on the numerical simulation of engineering problem. In this paper, by means of the maximum circle tensile stress on the curve of constant $\sigma$ and the stress intensity factors by a path independent contour integral method, trajectories of maxed mode crack propagation are simulated through numerical manifold method. The crack propagation is simulated dynamically by modifying the neighboring connection between the crack top and joint within an element. This method has the advantages such as less modified area, higher realizability and so on. Then a single sharp nicked specimen is used to verify the numerical results. It is shown that the proposed method is reasonable and effective.

2 Fundamentals of numerical manifold method

Numerical manifold method (NMM) proposed by Shi is derived from the finite covers theory in modern mathematics, and provides a unified framework for solving engineering calculation problem. NMM employs both a mathematical mesh and a physical mesh independently each other. The mathematical mesh is indicated by the computational consideration, while the physical mesh is constituted by the material boundary of the body of the engineering problem. The physical mesh divides mathematical mesh in physical covers which are interrelated through the application of
weighting function formulate. Then the finite-cover system covering the whole material field completely is formed to formulate this problem [2]. Covers formed from triangular meshes are shown in Fig. 1. The cover displacement functions \( u_i(x, y) \), \( v_i(x, y) \) are independently defined on individual physical cover \( C_i \). Local displacement functions can be connected through weighting functions \( w_i(x, y) \) to form a global displacement function on the whole material field, which is

\[
\begin{align*}
\{u(x, y)\} &= \sum_{i=1}^{4} w_i(x, y) \times \{u_i(x, y)\}, \quad (x, y) \in E \\
\{v(x, y)\} &= \sum_{i=1}^{4} w_i(x, y) \times \{v_i(x, y)\}, \quad (x, y) \in E
\end{align*}
\] (1)

where \( w_i(x, y) \) is the cover weighting function defined as follows.

\[
\begin{align*}
w_i(x, y) &= 0, \quad (x, y) \in C_i \\
w_i(x, y) &= 0, \quad (x, y) \notin C_i \\
\sum_{i=1}^{4} w_i(x, y) &= 1, \quad (x, y) \in E
\end{align*}
\] (2)

[Fig. 1 Covers formed from triangular meshes]

Displacement functions, \( u_i(x, y) \) and \( v_i(x, y) \), defined on physical cover \( C_i \), can be constant, linear, high order polynomial. In this paper, linear approximation is used.

\[
\begin{bmatrix}
d_{11} \\
d_{12} \\
d_{21} \\
d_{22}
\end{bmatrix}
\begin{bmatrix}
u_i(x, y) \\
v_i(x, y)
\end{bmatrix}
= [S] \cdot \{D_i\}
\] (3)

where \( d_{ij} \) is the displacement at node \( j \) on physical cover \( C_i \). \( \{D_i\} \) represents the displacement vector of physical cover \( C_i \). And there is

\[
[S] = \begin{bmatrix}
1 & 0 & x & 0 & y & 0 \\
0 & 1 & 0 & x & 0 & y
\end{bmatrix}
\]

The global displacement function on manifold elements is obtained below.

\[
\begin{align*}
\{u(x, y)\} &= \sum_{i=1}^{4} w_i(x, y) \{u_i(x, y)\} \\
\{v(x, y)\} &= \sum_{i=1}^{4} w_i(x, y) \{v_i(x, y)\}
\end{align*}
\]

where \( \{T\} = [w_1(x, y)S \ldots w_4(x, y)S] \) is the displacement matrix, and freedom vector \( \{D\} \) is defined by \( \{D\} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} \).

In a similar way to the formulation of the finite-element method, the stress, the strain matrix and control equation can be derived.

3 Crack propagation criteria of maximum circle tensile stress on counter \( \omega \) curves

Strain energy density describes the distribution of energy stored in micro-units when a component is subjected to external loads. There are curves of constant strain energy density like counter line around crack tip. Crack propagation is likely to occur in the direction that maximizes the circumferential stress as soon as strain energy density in units is equal [3].

Strain energy density \( \omega \) can be expressed by the following equation.

\[
\omega = \frac{S}{r}
\] (5)

where \( S \) is the strain energy density factor to describe the strength degree of energy field surrounding the crack tip, and is given by

\[
S = a_{i}K_i^2 + 2a_{12}K_1K_2 + a_{22}K_2^2 + a_{33}K_3^2
\] (6)

in which

\[
\begin{align*}
a_{i} &= \frac{1}{16\pi \mu}(1 + \cos \theta)(k - \cos \theta) \\
a_{12} &= \frac{1}{16\pi \mu}\sin \theta(2\cos \theta - k + 1) \\
a_{22} &= \frac{1}{16\pi \mu}\left[(k + 1)(1 - \cos \theta) + (1 + \cos \theta)(3\cos \theta - 1)\right] \\
a_{33} &= \frac{1}{4\pi \mu}
\end{align*}
\] (7)

where \( \mu \) is the shear modulus of material, \( k \) is a function of the Poisson’s ratios given by \( k = 3 - 4\nu \) in plane strain condition and \( k = \frac{3 - \nu}{1 + \nu} \) in plane stress condition. Let \( \omega \) equals constant \( A \), there is
which depicts a series of closed curves in polar coordinates of the crack tip, and these curves are called counter \( \omega \) curves. The crack propagation criteria of maximum circle tensile stress on counter \( \omega \) curves assumes that the crack propagation is likely to occur in the direction that maximizes the circumferential stress on counter \( \omega \) curves.

For a I-II mixed-mode crack, the loads are the stretch-loading normal to the crack plane and the shear-loading correspond to the crack plane, as is shown in Fig. 2. By using the analytic solution of linear-elastic fracture for the crack problem, the circumferential stress \( \sigma_{\omega \theta} \) of a I-II mixed-mode crack near the crack tip can be obtained as follows.

\[
\sigma_{\omega \theta} = \frac{K_{\omega 1}}{2\sqrt{2\pi}r} [K_{\omega 1}(1+\cos\theta) - K_{\omega 1}(3\sin\theta)] \cos\frac{\theta}{2}
\]

where \( \theta \) is the fracture angle of the propagation direction from the tangent to the crack path, which is determined by the solution that maximizes the function \( f(\theta) \) in the following equation

\[
\frac{\partial f(\theta)}{\partial \theta} = 0
\]

where

\[
f(\theta) = \frac{1}{\sqrt{S}} \left[ K_{\omega 1}(1+\cos\theta) - K_{\omega 1}(3\sin\theta) \right] \cos\frac{\theta}{2}
\]

Accurate numerical computation of stress intensity factors is critical in the numerical simulation of crack propagation by manifold method. Due to the irregularity of a crack propagation path under complex loading condition, in this paper, a contour integral method introduced in Ref. [4] is used to compute the stress intensity factors around the crack tip.

Considering mixed-mode crack in a planar body, two state solutions, i.e. the actual and the auxiliary solutions are obtained. Then we get the following equations according to Betti’s reciprocal work theorem.

\[
\int_{\Gamma} (u_{\omega} \dot{u}_{\omega} - \dot{u}_{\omega} u_{\omega}) d\Gamma = 0
\]

where \( \Gamma \) is the counter surrounding any zone near the crack tip, \( u_{\omega} \) and \( t_{\omega} \) are the distance and stress of the boundary \( \Gamma \) under the balance condition, while \( \dot{u}_{\omega} \) and \( \dot{t}_{\omega} \) are the distance and stress of the boundary \( \Gamma \) under the auxiliary balance condition. Given stress-free status on crack surface, the above integral along the closed counter as shown in Fig. 3 can be performed as follows.

\[
\int_{\Gamma_{1}} (u_{\omega} \dot{u}_{\omega} - \dot{u}_{\omega} u_{\omega}) d\Gamma = \int_{\Gamma_{2}} (u_{\omega} \dot{u}_{\omega} - \dot{u}_{\omega} u_{\omega}) d\Gamma
\]

If the counter \( \Gamma_{1} \) is defined as the circle of radius \( \delta \), wherein \( \delta \rightarrow 0 \), and \( \Gamma_{2} \) is set on a counter beyond the crack tip, \( u_{\omega} \) and \( t_{\omega} \) are the actual solutions on the counter and \( \dot{u}_{\omega} \) and \( \dot{t}_{\omega} \) the auxiliary ones, then denoting the integral on the counter \( \Gamma_{1} \) by \( m_{1}c_{1} + m_{2}c_{2} \) gives

\[
\frac{k+1}{2G} (K_{\omega 1}c_{1} + K_{\omega 2}c_{2}) = m_{1}c_{1} + m_{2}c_{2}
\]

The stress intensity factors of mixed-mode crack can be determined through comparison between the coefficients of \( c_{1} \) and \( c_{2} \).
and Possion’s ratio are $E = 3 \times 10^5$ MPa and $\mu = 0.3$, respectively.

$$E = 3 \times 10^5 \text{ MPa} \quad \mu = 0.3$$

Fig. 4 Specimen of infinite plate with a single prefab edge crack

The triangular mesh is used as the mathematical mesh, the specimen structure boundary containing a crack is defined as the physical mesh. The superposition of the physical mesh upon the mathematical mesh regenerates manifold covers of the solution field. There are 350 vertices and 614 physical covers as shown in Fig. 5, and the computing flowchart of numerical manifold method is shown in Fig. 6.

5.2 Meshing tracing technique

Under the distance boundary condition, the driver of crack propagation will reduce generally with the increasing of crack length subjected to determined external loads, and finally the crack propagation will stagnate completely when the driver is less than a critical value. If continuous propagation of the crack is maintained, it is necessary to increase the external loads to satisfy the propagation condition of the crack tip in the simulation process of crack propagation.

Consider a crack under external loads $P_s$, at the moment the crack is driven to position $l_{i,j}$ along the direction, which is determined according to numerical results in last computing cycle. The next step is to determine whether the crack is going to propagate in the original direction or not, if not, along which direction. Then it is required to compute SIF (stress intense factor) and compare the equivalent SIF with the critical value. If the equivalent SIF of the crack is larger than the critical value, it means that the crack will still propagate along the original direction determined in previous step, then an increment of crack propagation along the original direction is increased, and a new vertex $l_{i,j+1}$ is obtained. The geometry of mesh should be modified to accommodate the new crack configuration. We consider the standard processing operation as ill in the following case. First, the vertex $l_{i,j}$, which is the crack tip in the previous computing cycle, and the links, which are formed by connecting the vertex $l_{i,j}$ with the vertices of manifold element covering the vertex $l_{i,j}$ are removed. A new propagation crack increment is obtained, which institute a portion of physical boundary. The new crack tip $p_{i,j}$ is a template vertex of manifold element, which will connect with vertices of manifold element to create three new sub-element, as is shown in Fig. 7.

Fig. 5 Physical cover of specimen

Fig. 6 Computing flowchart of numerical manifold method

Fig. 7 Elements remeshing when common propagation of crack
If the efficient SIF is less than the critical value, the crack will stop propagating under the external loads $P_i$, and it is required to increase the external loads to keep the propagation condition of the crack tips, then the angle of the crack propagation should be computed. Because the new propagation direction may be different from the original direction, $l_{i,j}$ becomes curve vertex of the crack, which is a vertex of physical mesh. It is necessary to increase the external loads to drive the crack propagate continuously, extend the crack by a increment along the new propagation direction, and repeat the whole computing process, as is shown in Fig. 8. It is noted that the first increment of crack propagation after the curve vertex must be large enough to avoid forming an ill-element, as is shown in Fig. 9.

5.3 Mesh-Processing Method of ill-Element

The increment of crack propagation changes the physical boundary, while it becomes a part of physical mesh. it is necessary to re-divide the mathematic cover concerned. The standard partition is shown in Fig. 7, where the crack tip is connected directly with the vertex of the element in which crack tip lays to form new sub-element. However, the crack tip propagate near the vertex or link, and the element formed by the crack tip connected directly with the vertex of the element possibly generate ill-elements. It is required to modify locally the physical cover to accommodate the new propagating crack. The strategy used in this paper is to remove the links or vertices forming ill-elements, as shown in Figs. 10 to 13.

Connecting the crack tip with element vertices directly will produce an ill-element when the crack approach to a link, and removing the link and reconnecting the crack tip with vertices of manifold element will avoid forming ill-element, as can be seen in Figs. 10 to 13. Figs. 10 and 11 show the strategy of modifying mesh when the crack propagates straitly. Fig. 10 is the operation of processing mesh as the crack propagates to a link. When the crack-top propagates close to a link (Fig. 10a), the first operation is to remove the link that constitutes an ill-element (Fig. 10b) and then to connect the crack top and other neighboring vertex to form a new link (Fig. 10c), which will not constitute any ill-element. Fig. 11 shows the details of operation when the crack approaches to a vertex rectilinearly. When the crack-tip propagates near a vertex, the first operation is removing the vertex and the corresponding links, then connecting the crack tip and other neighboring verteae to form new elements. Figs. 12 and 13 show the strategies of modifying mesh when the crack propagate circuitously. Fig. 12 shows the details of operation when the crack-tip approach circuitously to a link. After a crack propagation curves, the curving point is kept as a vertex of the mesh, and the first crack increment after curving should be large enough in order to avoid generating an ill-element. If there is a link close to a vertex, the same strategy of modifying mesh is applied. Fig. 13 shows the details of operation when the crack approach to a vertex after curving.

Fig. 14 illustrates the path of crack propagation after limited steps of loading; the numerical simulation obtained in this work is similar to the results in Ref. [10].
6 Conclusions

The numerical manifold method (NMM) is a new numerical method; it can be considered as a union of generalized finite element method and discontinuous deformation analysis, and it provides a unified framework for solving general engineering problems dealing with both continuous and discontinuous materials. The most innovative feature of NMM is that it employs both physical mesh and mathematical mesh to formulate the physical problem. The mathematical mesh is indicated by the computational consideration, and the physical mesh is constituted by the material boundary involving the crack. It is convenient to realize programming when applying NMM to studying crack propagation behaviors; it does not require continually remeshing entirely after each crack increment at a variety of loads. In this paper, NMM combined with the maximum circle tensile stress on constant $\omega$ curves and stress intensity factors by a path independent contour integral method is proposed to simulate the mixed mode crack propagation. The crack propagation is traced dynamically by modifying the neighboring connection between the crack-top and nodes within the element in the calculating process to avoid the ill-element. A single sharp nicked specimen is used to verified the numerical results. It is shown that the proposed method is reasonable and effective.

References