Entire functions sharing one small function

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Abstract: The uniqueness problem of entire functions sharing one small function was studied. By Picard's Theorem, we proved that for two transcendental entire functions \( f(z) \) and \( g(z) \), a positive integer \( n \geq 9 \), and \( a(z) \) (not identically equal to zero) being a common small function related to \( f(z) \) and \( g(z) \), if \( f^n(z)(f(z)-1)g'(z) \) and \( g^n(z)(g(z)-1)g'(z) \) share \( a(z) \) CM, where CM is counting multiplicity, then \( g(z) = f(z) \). This is an extended version of Fang and Hong's theorem [Fang ML, Hong W, A unicity theorem for entire functions concerning differential polynomials, Journal of Indian Pure Applied Mathematics, 2001, 32 (9): 1343-1348].

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1 Introduction

Let \( f(z) \) be a non-constant meromorphic function in the whole complex plane. We use the standard notations of value distribution theory [1] including \( T(r,f) \), \( m(r,f) \), \( N(r,f) \), \( \overline{N}(r,f) \), etc., and define \( S(r,f) \) by \( S(r,f) = o(T(r,f)) \) as \( r \to +\infty \), possibly outside a set with finite measure.

Let \( a(z) \) be a meromorphic function. If \( T(r,a) = S(r,f) \), then \( a(z) \) is called a small function related to \( f(z) \).

Let \( a \) be a finite complex number. We denote by \( N_2(r, \frac{1}{f-a}) \) the counting function for zeros of \( f(z) - a \) with multiplicity not more than 2, and by \( \overline{N}_2(r, \frac{1}{f-a}) \) the corresponding function for those whose multiplicities are not counted. Let \( N_c(r, \frac{1}{f-a}) \) be the counting function for zeros of \( f(z) - a \) with multiplicity at least 2 and \( \overline{N}_c(r, \frac{1}{f-a}) \) the corresponding function for those whose multiplicities are not counted. Set \( N_2(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_c(r, \frac{1}{f-a}) \).

Let \( g(z) \) be a meromorphic function, and \( a(z) \) a common small function related to \( f(z) \) and \( g(z) \). If \( f(z) - a(z) \) and \( g(z) - a(z) \) assume the same zeros with the same multiplicities, then we call that \( f(z) \) and \( g(z) \) share the small function \( a(z) \) CM, where CM is counting multiplicity.

Fang and Hua [2-3] proved the following results.

**Theorem A** [2]. Let \( f(z) \) and \( g(z) \) be two transcendental entire functions, and \( n \geq 6 \) be a positive integer. If \( f^n(z)f'(z) \) and \( g^n(z)g'(z) \) share 1 CM, then either \( f^nfg^n \equiv 1 \) or \( g \equiv cf \) for a constant \( c \) with \( c^{n+1} \equiv 1 \).

**Theorem B** [3]. Let \( f(z) \) and \( g(z) \) be two transcendental entire functions, and \( n \geq 6 \) be a positive integer. Assume that \( a(z) \neq 0 \) is a common small function related to \( f(z) \) and \( g(z) \). If \( f^n(z)f'(z) \) and \( g^n(z)g'(z) \) share \( a(z) \) CM, then for a constant \( c \) with \( c^{n+1} \equiv 1 \), either \( f^nfg^n \equiv [a(z)]^c \) or \( g \equiv cf \).

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Fang and Hong [4] proved the following result.

**Theorem C** [4]. Let \( f(z) \) and \( g(z) \) be two transcendental entire functions, and \( n \geq 11 \) be a positive integer. If \( f''(z)(f(z) - 1)f'(z) \) and \( g''(z)(g(z) - 1)g'(z) \) share \( 1 \) \( CM \), then \( g(z) \equiv f(z) \).

Our purpose in this study was to prove that there exists the following extended solution to Theorem C.

**Theorem 1.** Let \( f(z) \) and \( g(z) \) be two transcendental entire functions, and \( n \geq 9 \) be a positive integer. Assume that \( a(z) \equiv 0 \) is a common small function related to \( f(z) \) and \( g(z) \). If \( f''(z)(f(z) - 1)f'(z) \) and \( g''(z)(g(z) - 1)g'(z) \) share \( a(z) \) \( CM \), then \( g(z) \equiv f(z) \).

## 2 Lemmas

To prove Theorem 1, we first introduce the following lemmas.

**Lemma 1** [5]. Let \( f(z) \) be a meromorphic function. Then,

\[
T(r, a, f'' + a_{-1}f' + \cdots + a_1f + a_0) = nT(r, f) + S(r, f)
\]

where \( a_0(\neq 0), a_{-1}, \ldots, a_0 \) are constants.

**Lemma 2** [5]. Let \( f_k(z)(k = 1, 2, \ldots, n) \) be linearly independent meromorphic functions, and \( n \) is a positive integer. If \( \sum_{k=1}^{n} f_k(z) = 1 \) and \( \sum_{k=1}^{n} N(r, f_k) = S(r) \), then,

\[
T(r) \leq \sum_{k=1}^{n} N(r, \frac{1}{f_k}) - N(r, \frac{1}{D}) + S(r),
\]

where \( D(f_1, f_2, \ldots, f_n) \) is the Wronskian determinant of \( f_k(z)(k = 1, 2, \ldots, n); \) and \( S(r) = o[T(r)] \) \( r \to +\infty, \) \( r \notin E \). Here,

\[
T(r) = \max_{1 \leq k \leq n}[T(r, f_k)],
\]

and \( E \) is a set of finite measure.

**Lemma 3** [5]. Let \( f_k(z)(k = 1, 2, 3) \) be meromorphic functions in the whole complex plane, and \( f_1(z) \) be a non-constant meromorphic function. If \( f_k(z)(k = 1, 2, 3) \) are linearly dependent meromorphic functions, then, \( f_1(z) + f_2(z) + f_3(z) = 1 \), and

\[
\overline{N}(r, \frac{1}{f_1}) + \overline{N}(r, \frac{1}{f_2}) + \overline{N}(r, f_3) \leq (\lambda + o(1))T(r).
\]

Here,

\[
0 < \lambda < 1, \quad T(r) = \max_{1 \leq k \leq 3}[T(r, f_k)]
\]

Then, \( f_2 \equiv 1 \), or \( f_3 \equiv 1 \).

**Lemma 4** [5]. Let \( f(z) \) be a non-constant meromorphic function in the whole complex plane, and \( k \) be a positive integer. Then,

\[
N(r, \frac{1}{f}) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f).
\]

## 3 Proof of Theorem 1

By the assumption of Theorem 1, we have

\[
\frac{f''(f - 1)f' - a(z)}{g''(g - 1)g' - a(z)} = e^{h(z)},
\]

where \( h(z) \) is an entire function. Thus,

\[
\frac{f''(f - 1)f'}{a(z)} - \frac{e^{h(z)}g''(g - 1)g'}{a(z)} + e^{h(z)} = 1.
\]

Then,

\[
N(r, \frac{1}{f''(f - 1)f')} \leq 2N(r, \frac{1}{f}) + N(r, \frac{1}{f}) + N(r, \frac{1}{f'}) + 3N(r, \frac{1}{f}) + 4T(r, \frac{1}{f}) + S(r, f),
\]

where \( D(f_1, f_2, \ldots, f_n) \) is the Wronskian determinant of \( f_k(z)(k = 1, 2, \ldots, n); \) and \( S(r) = o[T(r)] \) \( r \to +\infty, \) \( r \notin E \). Here,

\[
T(r) = \max_{1 \leq k \leq n}[T(r, f_k)],
\]

and \( E \) is a set of finite measure.

**Lemma 3** [5]. Let \( f_k(z)(k = 1, 2, 3) \) be meromorphic functions in the whole complex plane, and \( f_1(z) \) be a non-constant meromorphic function. If \( f_k(z)(k = 1, 2, 3) \) are linearly dependent meromorphic functions, then, \( f_1(z) + f_2(z) + f_3(z) = 1 \), and

\[
\overline{N}(r, \frac{1}{f_1}) + \overline{N}(r, \frac{1}{f_2}) + \overline{N}(r, f_3) \leq (\lambda + o(1))T(r).
\]

Here,

\[
0 < \lambda < 1, \quad T(r) = \max_{1 \leq k \leq 3}[T(r, f_k)]
\]

Then, \( f_2 \equiv 1 \), or \( f_3 \equiv 1 \).

**Lemma 4** [5]. Let \( f(z) \) be a non-constant meromorphic function in the whole complex plane, and \( k \) be a positive integer. Then,

\[
N(r, \frac{1}{f}) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f).
\]

By the assumption of Theorem 1, we have

\[
\frac{f''(f - 1)f' - a(z)}{g''(g - 1)g' - a(z)} = e^{h(z)},
\]

where \( h(z) \) is an entire function. Thus,

\[
\frac{f''(f - 1)f'}{a(z)} - \frac{e^{h(z)}g''(g - 1)g'}{a(z)} + e^{h(z)} = 1.
\]

Then,

\[
N(r, \frac{1}{f''(f - 1)f')} \leq 2N(r, \frac{1}{f}) + N(r, \frac{1}{f}) + N(r, \frac{1}{f}) + 3N(r, \frac{1}{f}) + 4T(r, \frac{1}{f}) + S(r, f),
\]

where \( D(f_1, f_2, \ldots, f_n) \) is the Wronskian determinant of \( f_k(z)(k = 1, 2, \ldots, n); \) and \( S(r) = o[T(r)] \) \( r \to +\infty, \) \( r \notin E \). Here,

\[
T(r) = \max_{1 \leq k \leq n}[T(r, f_k)],
\]

and \( E \) is a set of finite measure.
and from Expressions (3) and (5),
\[
N_2(r, \frac{1}{f''(f-1)f'}) \leq \frac{4}{n} T(r, \frac{a(z)}{f''(f-1)f'}) + S(r, f).
\] (6)

Likewise, there exists
\[
N_2(r, \frac{1}{e^{(z)}g''(g-1)g'}) \leq \frac{4}{n} T(r, \frac{e^{(z)}g''(g-1)g'}{a(z)}) + S(r, g).
\] (7)

We claim that
\[
\frac{f''(f-1)f'}{a(z)}, \; \frac{-e^{(z)}g''(g-1)g'}{a(z)},
\]
and \(e^{(z)}\) are linearly dependent meromorphic functions.

Suppose that
\[
\frac{f''(f-1)f'}{a(z)}, \; \frac{-e^{(z)}g''(g-1)g'}{a(z)},
\]
and \(e^{(z)}\) are linearly independent meromorphic functions.

Then,
\[
D(f''(f-1)f'), \; \frac{e^{(z)}g''(g-1)g'}{a(z)}, \; \frac{e^{(z)}}{a(z)} = \frac{f''(f-1)f'}{a(z)} \leq \frac{1}{n} T(r, \frac{e^{(z)}g''(g-1)g'}{a(z)}) + S(r) \leq \frac{8}{n} T(r) + S(r) \leq \frac{8}{9} T(r) + S(r),
\]
where \(q(z)\) is a differential polynomial in \(f, \; g, \; h(z), \text{ and } a(z)\). Thus, we can easily obtain
\[
N(r, \frac{1}{f}) + S(r) \geq (n-2)N(r, \frac{1}{f}) + (n-2)N(r, \frac{1}{g}) + N(r, \frac{1}{e^{(z)}}).
\]

By Lemma 2,
\[
T(r) \leq N(r, \frac{a(z)}{f''(f-1)f'}) + N(r, \frac{a(z)}{e^{(z)}g''(g-1)g'}) + N(r, \frac{1}{e^{(z)}}) - N(r, \frac{1}{f}) + S(r) \leq N(r, \frac{a(z)}{f''(f-1)f'}) + N(r, \frac{a(z)}{e^{(z)}g''(g-1)g'}) - N(r, \frac{1}{f}) + S(r) \leq 2N(r, \frac{1}{f}) + N(r, \frac{1}{f-1}) + N(r, \frac{1}{g}) + 2N(r, \frac{1}{g-1}) + N(r, \frac{1}{g}) + N(r, \frac{1}{g-1}) + S(r) \leq \frac{12}{n} T(r) + S(r),
\]
where \(D(f''(f-1)f'), \; \frac{e^{(z)}g''(g-1)g'}{a(z)}, \; \frac{e^{(z)}}{a(z)}\) is the Wronskian determinant of \(f''(f-1)f', \; \frac{e^{(z)}g''(g-1)g'}{a(z)}, \; \frac{e^{(z)}}{a(z)}\), and \(S(r) = o[T(r)], (r \rightarrow +\infty, r \not\in E)\).

Here,
\[
T(r) = \max \{T(r, \frac{f''(f-1)f'}{a(z)}), T(r, \frac{e^{(z)}g''(g-1)g'}{a(z)}),
\]

and \(E\) is a set of finite measure.

Thus, we get a contradiction. This proves that
\[
\frac{f''(f-1)f'}{a(z)}, \; \frac{-e^{(z)}g''(g-1)g'}{a(z)}, \text{ and } e^{(z)}\text{ are linearly dependent meromorphic functions.}
\]

From
\[
\frac{N(r, \frac{a(z)}{f''(f-1)f'}) + N(r, \frac{a(z)}{e^{(z)}g''(g-1)g'}) + N(r, \frac{1}{e^{(z)}})}{a(z)} \leq \frac{1}{f''(f-1)f') + N(r, \frac{1}{e^{(z)}g''(g-1)g'}) + \frac{1}{a(z)} + S(r) \leq \frac{8}{n} T(r) + S(r) \leq \frac{8}{9} T(r) + S(r),
\]
and by Lemma 3,
\[
\frac{-e^{(z)}g''(g-1)g'}{a(z)} \equiv 1, \text{ or } e^{(z)} \equiv 1.
\]

**Case 1.** \(- \frac{e^{(z)}g''(g-1)g'}{a(z)} = 1\). By it, \(g''(g-1)g' = \frac{a(z)}{e^{(z)}}\). From Identical Equation (2), \(f''(f-1)f' = -a(z)e^{(z)}\). Therefore, \(f''(f-1)f'g''(g-1)g' = [a(z)]^2\).

From the above and the conditions of Theorem 1, any zero or 1-point of \(f(z)\) must be a zero of \(a(z)\).

By the second fundamental theorem,
\[ T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, -\frac{1}{f-1}) + S(r, f) \leq N\left(\frac{1}{a}\right) + S(r, f), \]

which is a contradiction. It shows that \( f^*(f-1)f'g^*(g-1)g' \neq [a(z)]^2 \).

**Case 2.** \( e^{\frac{f(z)}{n}} \equiv 1 \). By Identical Equation (2), \( f^*(f-1)f' = g^n(g-1)g' \). Solving it gives
\[ f^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right) = g^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right) + c, \tag{8} \]
where \( c \) is constant. Set \( F = f^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right)f' \).

We claim that \( c = 0 \). If \( c \neq 0 \), then
\[ \frac{-\overline{N}(r, 0, f^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right))}{T(r, f^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right))} = \]
\[ \frac{-\overline{N}(r, 0, f) + \overline{N}(r, 0, f - \frac{n + 2}{n + 1})}{T(r, f^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right))} \leq \]
\[ 2T(r, f) + O(1) \leq \frac{2}{n + 2} - \frac{2}{11}. \tag{9} \]

Likewise,
\[ \frac{-\overline{N}(r, 0, g^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right))}{T(r, g^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right))} \leq \frac{2}{11}. \tag{10} \]

Thus, by (9) and (10),
\[ \Theta(0, F) + \Theta(c, F) = \Theta(0, f^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right)) + \]
\[ \Theta(0, g^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right)) \geq 2 - \frac{4}{11} > 1, \]
which contradicts \( \Theta(0, f^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right)) + \Theta(0, g^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right)) \leq 1 \). Thus,
\[ f^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right) = g^{*n+1}\left(1 + \frac{1}{n + 2} - \frac{1}{n + 1}\right). \tag{11} \]

Set \( G = f / g \). If \( G \not\equiv 1 \), by (11),
\[ g = \frac{(n + 2)(G^{*n+1}-1)}{(n + 1)(G^{*n+2}-1)}. \]

This completes the proof of Picard’s theorem that \( G \) is a constant. Hence, \( g \) is a constant, which is a contradiction. Therefore, \( G(z) \equiv 1 \), i.e. \( g(z) \equiv f(z) \).

**References**


