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Reliability analysis of a two-dependent-unit parallel system with a standby unit *

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Abstract: A parallel system with two active components and a cold standby unit is studied in this paper. The two simultaneously working components are dependent and the copula function is used to model their dependence. An explicit expression is obtained for the mean time to failure of the system in terms of the copula function and marginal lifetime distributions in two different cases. As an application, numerical calculations are presented corresponding to two different copula functions and marginal lifetime distributions.

Keywords: reliability; copula; parallel system; mean time to failure

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1 Introduction

We know that standby redundancy is a common and effective way to improve the reliability of an engineering system. In Lai and Xie (2003) [1], as well in other related studies of system reliability under standby redundancy, it is all assumed that the components are independent. But this assumption is rarely valid in practice. In fact, in reliability analysis, the component lifetimes are usually co-dependent. In most of real life engineering systems the dependence among the components is inevitable due to the common random production and operating environments. Two components in a reliability structure may share the same load or may be subject to the same stresses. In such a case, lifetimes of the two components will be related to each other, or to be co-dependent. Usually the failure time of the components tend to be longer or shorter at the same time, and this is a form of positive dependence. Most of the studies on systems with dependent components focus on the case when there is no standby component [2-6].

Copula function is essentially an n-dimensional distribution function on the n-dimensional unit cube with uniform margins, and it is an efficient tool for investigating the dependence structure between random variables [7-8]. In system reliability, the notion of copulas has been found to be useful for modeling dependence, and copulas are used to create a multivariate lifetime distribution for modeling dependence among the components [3-49]. The unit is said to be in the case of cold standby if it does not fail while in standby. When one or two of the working units fail, then a cold standby unit is immediately put into operation. Some recent discussions on systems with a standby component are in Refs. [10-12]. Reliability of two-dependent-unit series system with a standby unit was studied by Eryilmaz and Tank [13].

In this work, we study two-unit parallel systems with a single cold standby component and the two simultaneously working components are dependent through a given copula.

The paper is organized as follows. In section 2, we define the lifetime of the system and compute its
expected value, i.e. mean time to failure (MTTF) and bounds for the MTTF when the components are positively or negatively quadrant-dependent. Section 3 contains extensive numerical calculations for different copula functions and marginal lifetime distributions. Conclusion and some remarks are given in Section 4.

2 Lifetime and MTTF of the system

Consider a two-unit parallel system with one cold standby component. Let $X_i$ be the lifetime of the $i$th active component ($i = 1, 2$), $Z$ the lifetime of the standby component, and $T$ the lifetime of the parallel system. In the parallel system, the system fails when all components fail. Let $F_i(t)$ and $P(X_i > t) = 1 - F_i(t)$ be the distribution and survival functions of $X_i$ ($i = 1, 2$).

We assume that the system does not repair the components when it fails to work properly. The dependence between the components working in the common random environment is inevitable, and if the components are identical, it can be assumed that the distributions and survival functions of them are identical and the common random environment makes them exchangeably dependent. Obviously, the lifetime $Z$ of the standby component is independent and has an identical distribution to $X_1$. In this paper, we model this dependence by the exchangeable copula function $C$, i.e. $C(u, v) = C(v, u)$ for all $u, v \in [0, 1]$. In the following, we consider two different uses of the cold standby unit.

2.1 Case 1

If the cold standby unit is put into operation when the two simultaneously working components both fail, then the lifetime of the systems is represented as

$$T = \max\{X_1, X_2\} + Z.$$ 

Let $X_1$ and $X_2$ be exchangeably dependent with an absolutely continuous joint distribution and this dependence is modeled by the copula function $C(u, v)$, and they have the same distribution function $F(t)$. According to Sklar theorem, the joint cumulative distribution function (cdf) of $X_1$ and $X_2$ is given by

$$F(t_1, t_2) = P\{X_1 \leq t_1, X_2 \leq t_2\} = C(F(t_1), F(t_2)).$$

If $C(u, v)$ is exchangeable, then $F(t_1, t_2) = F(t_2, t_1)$.

To solve the expectation of $T$, we have

$$P\{\max(X_1, X_2) > t\} = 1 - P\{\max(X_1, X_2) \leq t\} = 1 - P(X_1 \leq t, X_2 \leq t) = 1 - C(F(t), F(t)) = 1 - C(F(t), F(t)).$$

So, the MTTF of the parallel system with a single cold standby unit is computed with

$$E(T) = E(\max(X_1, X_2)) + E(Z) = \int_0^\infty [1 - C(F(t), F(t))] dt + E(Z).$$

There are various notions of dependence. A pair of random variables $(X, Y)$ is said to be positively quadrant-dependent (PQD) if

$$P(X \leq t_1, Y \leq t_2) \geq P(X \leq t_1)P(Y \leq t_2)$$

for all $t_1$ and $t_2$, or equivalently

$$P(X > t_1, Y > t_2) \geq P(X > t_1)P(Y > t_2).$$

If the inequality in (1) is reversed, $(X, Y)$ is negatively quadrant-dependent (NQD). In terms of a copula function, the inequality (1) is equivalent to $C(u, v) \geq uv$ for all $u, v \in [0, 1]$. Similarly NQD is equivalent to $C(u, v) \leq uv$ for all $u, v \in [0, 1]$.

Assume that the cold standby component has the same marginal distribution with $X_1$, i.e. $P[Z \leq t] = F(t)$. We obtain the bound of $E(T)$ in the following theorem.

**Theorem 1.** If the lifetime $X_1$ and $X_2$ of two components are PQD, we can obtain that the upper bound of $E(T)$ is $\int_0^\infty [2 - F(t) - F^2(t)] dt$, and the lower bound of $E(T)$ is $\int_0^\infty [2 - 2F(t)] dt = 2E(X_1)$.

**Proof.** If $X_1$ and $X_2$ are PQD, then $C(F(t), F(t)) \geq F^2(t)$ . Hence we have

$$E(T) = \int_0^\infty [1 - C(F(t), F(t))] dt + E(Z) \leq \int_0^\infty [1 - F^2(t)] dt + \int_0^\infty [1 - F(t)] dt = \int_0^\infty [2 - F(t) - F^2(t)] dt.$$
Because \( C(u,u) \leq u \), we have \( C(F(t), F(t)) \leq F(t) \).

Thus,

\[
E(T) = \int_0^\infty [1 - C(F(t), F(t))] dt + E(Z) \geq \int_0^\infty [1 - F(t)] dt + \int_0^\infty [1 - F(t)] dt = 2E(X_i).
\]

**Remark 1.** The upper bound \( \int_0^\infty [2 - F(t) - F^2(t)] dt \) can be obtained if \( X_1 \) and \( X_2 \) are independent, i.e. \( C(u,v) = uv \).

**Remark 2.** The lower bound \( \int_0^\infty [2 - 2F(t)] dt = 2E(X_i) \) can be attained if \( C(u,v) = M = \min(u,v) \).

**Theorem 2.** If the lifetimes \( X_1 \) and \( X_2 \) of two components are NQD, the upper bound of \( E(T) \) is \( 3E(X_i) \), and the lower bound of \( E(T) \) is \( \int_0^\infty [2 - F(t) - F^2(t)] dt \).

**Proof.** If \( X_1 \) and \( X_2 \) are NQD, then \( C(F(t), F(t)) \leq F^2(t) \). Hence we have

\[
E(T) = \int_0^\infty [1 - C(F(t), F(t))] dt + E(Z) \geq \int_0^\infty [1 - F(t)] dt + \int_0^\infty [1 - F(t)] dt = 3E(X_i).
\]

Since \( C(1,1) = \min(u,v) \leq [1-u] + [1-v] \), we obtain

\[
1 - C(F(t), F(t)) \leq (1 - F(t)) + (1 - F(t)).
\]

Then,

\[
E(T) = \int_0^\infty [1 - C(F(t), F(t))] dt + E(Z) \leq \int_0^\infty [2(1 - F(t))] dt + \int_0^\infty [1 - F(t)] dt = 3E(X_i).
\]

**Remark 3.** The lower bound \( \int_0^\infty [2 - F(t) - F^2(t)] dt \) can be attained if \( X_1 \) and \( X_2 \) are independent, i.e. \( C(u,v) = uv \).

2.1 Case II

If the cold standby unit is put into operation when only one working unit fails, then the lifetime of the system is

\[
T = \min(X_i, X_j) + \max(X_i^*, Z),
\]

where \( X_i^* \) is the residual lifetime of surviving active component after the first failure in the system, i.e.

\[
X_i^* = \min(X_i, X_j)\{X_i > \min(X_i, X_j)\}.
\]

Under these assumptions, \( X_1 \) and \( X_2 \) are exchangeable and the cdf of the random variable \( X_1^* \) is as given in Ref. [13].

**Proposition 1.** \(^{(13)}\) If \( X_1 \) and \( X_2 \) are exchangeable and have an identical distribution function \( F(x) \) and \( X_1^* \) is defined as above, the distribution of \( X_1^* \) is

\[
F(t) = 2\int_0^t \int_0^t c(F(x), F(y)) dF(x) dF(y) - 1 = 2\int_0^t \int_0^t c(F(x), F(y)) dF(x) dF(y),
\]

(2)

where \( t \geq 0 \) and \( c(u,v) = \frac{\partial^2}{\partial u \partial v} C(u,v) \).

Assume that the cold standby component has the same marginal distribution with \( X_i \), i.e. \( P[Z \leq t] = F(t) \). After the first failure in the system, the standby component is put into operation and it works together with the remaining component whose cdf is given by Eq. (2); thus the joint distribution of \( X_i^* \) and \( Z \) is no longer exchangeable. They are again assumed to be dependent with the joint cdf:

\[
P[X_i^* \leq t_1, X_2 \leq t_2] = C(F^*(t_1), F(t_2)) \quad \text{for } t_1, t_2 \geq 0.
\]

The joint survival function of \((X_1, X_2)\) is given by

\[
P\{X_i > t_1, X_2 > t_2\} = 1 - P(X_i < t_1) - P(X_2 < t_2) + P(X_i < t_1, X_2 < t_2) = 1 - F(t_1) - F(t_2) + C(F(t_1), F(t_2)).
\]

(3)

Thus using Eq. (3), the survival functions of \( \min(X_1, X_2) \) and \( \max(X_1, X_2) \) are found to be

\[
P\{\min(X_1, X_2) > t\} = P(X_i > t, X_2 > t) = 1 - 2F(t) + C(F(t), F(t)),
\]

(4)
and

\[ P[\max(X_i^*, Z) > t] = 1 - C(F^*(t), F(t)). \]  \hfill (5)

Using these survival functions, the MTTF of the parallel system with a single cold standby unit is given by the following theorem.

**Definition 1.** The nonnegative random variable \( X \) is NBU (new better than used) if and only if \( X \geq \alpha \left[ X - t \right] \) for all \( t \geq 0 \), it is NWU (new worse than used) if and only if \( X \leq \alpha \left[ X - t \right] \) for all \( t \geq 0 \).

**Theorem 3.** Let \( X_1 \) and \( X_2 \) have the identical distribution function \( F(x), X_1 \) and \( X_2 \) are NBU; \( X_i^* \) is defined as above and has cdf \( F^*(x) \). Then the expectation of

\[ T = \min(X_1, X_2) + \max(X_i^*, Z) \]

is

\[ E(T) = \int_0^\infty [1 - 2F(t) + C(F(t), F(t))] dt + \int_0^\infty [1 - C(F^*(t), F(t))] dt, \]

the upper bound of \( E(T) \) is \( \int_0^\infty [2 - 2F(t)] dt = 2E(X_i) \)

and the lower bound of \( E(T) \) is \( \int_0^\infty (1 - F(t)) dt = E(X_i) \).

**Proof.** Because \( T = \min(X_1, X_2) + \max(X_i^*, Z) \) and we have

\[ E(T) = \bar{E} \{ \min(X_1, X_2) \} + \bar{E} \{ \max(X_i^*, Z) \}. \]

Obviously, based n Eqs. (4) and (5), we obtain

\[ E(T) = \int_0^\infty [2 - 2F(t) + C(F(t), F(t)) - C(F^*(t), F(t))] dt. \]

As is mentioned above, \( X_1 \) and \( X_2 \) are NBU. From Definition 1 we can prove that

\[ \bar{F}^*(t) = P(X_1^* > t, X_2 > t) \leq P(X_1 > t) = \bar{F}(t). \]

Then,

\[ F^*(t) \geq F(t), \]

\[ C(F(t), F(t)) \leq C(F^*(t), F(t)). \]

We get

\[ E(T) = \int_0^\infty [2 - 2F(t) + C(F(t), F(t)) - C(F^*(t), F(t))] dt \leq \int_0^\infty [2 - 2F(t)] dt = 2E(X_i). \]

From \( C(F(t), F(t)) - C(F^*(t), F(t)) \geq F(t) - F^*(t) \),

we obtain

\[ E(T) = \int_0^\infty [2 - 2F(t) + C(F(t), F(t)) - C(F^*(t), F(t))] dt \geq \int_0^\infty (1 - F(t)) dt = E(X_i). \]

**Remark 4.** The upper bound \( \int_0^\infty [2 - 2F(t)] dt = 2E(X_i) \) can be attained if \( C(u, v) = M = \min(u, v) \).

It should be noted that similar bounds have been obtained in Ref. [14] for the two-unit parallel system without a standby.

3 **Illustrations**

Farlie-Gumbel-Morgenstern (FGM) family and Ali-Mikhail-Haq family are the totally ordered one-parameter families of copulas that have subfamilies of PQD copulas and NQD copulas. These model includes independence for \( \alpha = 0 \) and has PQD (NQD) property when \( \alpha \geq \alpha = 0 \).

First assume that the dependence between the components is modeled by FGM copula given by

\[ C_4(u, v) = \alpha v u + \alpha v (1 - u)(1 - v) \]

for \( -1 \leq \alpha \leq 1 \) and \( u, v \in [0, 1] \).

It is easy to see that,

\[ c_4(u, v) = \frac{\partial^2}{\partial udv} C_4(u, v) = 1 + \alpha (1 - 2u)(1 - 2v). \]  \hfill (6)

Let the marginal distribution \( F \) be exponential with cdf \( F(t) = 1 - e^{-ut}, t \geq 0 \), Eryilmaz and Tank [13] showed that

\[ F^*(t) = 1 - e^{-ut} + \frac{\alpha}{3} e^{-ut}(1 - e^{-ut}) \]

for \( t \geq 0 \).
It should be noted that when \( \alpha = 0 \), i.e. \( X_1 \) and \( X_2 \) are independent, we have \( F^*(t) = 1 - e^{-\lambda t} \), \( t \geq 0 \). That is, the residual lifetime distribution of surviving component does not change and it has again an exponential distribution with parameter \( \lambda \). This is a direct consequence of memoryless property of exponential distribution. For the exponential marginal distribution, the survival function of \( \min(X_1, X_2) \) is
\[
P[\min(X_1, X_2) > t] = 1 - F(t) + C(F(t), F(t)) = (1 + \alpha) e^{-\lambda t} - 2\alpha e^{-\lambda t} + \alpha e^{-2\lambda t} \text{ for } t \geq 0.
\] (7)

On the other hand, the survival function of \( \max(X_1, Z) \) is
\[
P[\max(X_1, Z) > t] = 1 - C(F(t), F(t)) = (2\alpha - 2) e^{-\lambda t} + (1 - \alpha) e^{-\lambda t} + \frac{5\alpha}{3} - \frac{4\alpha^2}{3} + \frac{\alpha^3}{2} - \frac{\alpha^3}{9} e^{-\lambda t} - \frac{2\alpha^3}{3} - \frac{\alpha^3}{9} e^{-6\lambda t}.
\]

About Case I in Section 2, the MTTF of the system is
\[
E(T) = \int_0^\infty [1 - C(F(t), F(t))] dt + \int_0^\infty [1 - F(t)] dt = \frac{1}{\lambda} \left( -\frac{1}{2} - \frac{1}{12}\alpha \right).
\]

When we consider MTTF0 without a standby component in the parallel system, i.e. \( E(\max(X_1, X_2)) \)
\[
E(T) = E(\max(X_1, X_2)) = \int_0^\infty [1 - C(F(t), F(t))] dt = \frac{1}{\lambda} \left( -\frac{1}{2} - \frac{1}{12}\alpha \right).
\]

About Case II in Section 2, the MTTF of the system is
\[
E(T) = \int_0^\infty [1 - 2F(t) + C(F(t), F(t))] dt + \int_0^\infty [1 - C(F(t), F^*(t))] dt = \frac{1}{\lambda} \left( -\frac{7}{36}\alpha + \frac{1}{180}\alpha^2 + \frac{1}{540}\alpha^3 \right), \quad -1 \leq \alpha \leq 1.
\]

Next assume that the dependence between the components is modeled by Ali-Mikhail-Haq copula given by
\[
C_2(u, v) = \frac{uv}{1 - \alpha((1-u)(1-v))}, \quad -1 \leq \alpha \leq 1, \quad u, v \in [0,1].
\]

For the case \( \alpha = 0 \) we obtain an independence copula. The joint density associated with \( C_2(u, v) \) is
\[
C_2(u, v) = \frac{(1 - \alpha)^2 + \alpha^2(2uv - u - v) + \alpha(u + v + uv)}{[1 - \alpha((1-u)(1-v))]^3}.
\]

Since \( C_2(u, v) \) has a complicated form, it is not easy to get an exact analytical expression for \( E(T) \). However, it can be computed by means of numerical integration and we consider \( \alpha = 1 \).

The bivariate models considered in reliability theory are mostly PQD. In Table 1, we compute the MTTF0 of the system for FGM copula and the MTTF1 of the system for Ali-Mikhail-Haq when the marginal distributions of the components are exponential with unit mean. This Table also includes the MTTF0 without a standby component in the parallel system, i.e. \( E(\max(X_1, X_2)) \) for FGM ( MTTF0 ) copula and Ali-Mikhail-Haq ( MTTF1 ) copula. On the other hand, we observe that the MTTF value under FGM copula is greater than the MTTF under Ali-Mikhail-Haq copula when there is no standby component in the parallel system.

These can be explained as follows. For two copulas \( C_1 \) and \( C_2 \), \( C_1 \) is said to be less positively dependent than a copula \( C_2 \) (written as \( C_1 \preceq_{\text{PQD}} C_2 \)) if \( C_1(u, v) \leq C_2(u, v) \) for all \( u, v \in [0,1] \). We have the following relations between Ali-Mikhail-Haq and FGM copulas (see Ref. [7]).

\[
C_2(u, v) = uv[1 + \alpha((1-u)(1-v)) + \alpha \sum_{r=2}^\infty \alpha((1-u)(1-v))^r] = C_1(u, v) + uv \sum_{r=2}^\infty \alpha((1-u)(1-v))^r \geq C_1(u, v).
\]

This implies that the FGM copula is less positively dependent than the Ali-Mikhail-Haq copula. That is, if \( C_1 \preceq_{\text{PQD}} C_2 \), then
\[
\text{MTTF}_0 = E(\max(X_1, X_2) | C_1) = \int_0^\infty (1 - C_1(F(t), F(t))) dt \geq \int_0^\infty (1 - C_2(F(t), F(t))) dt = E(\max(X_1, X_2) | C_2) = \text{MTTF}_1.
\]
From Table 1, we can find that the MTTF for \( \alpha > 0 \) is smaller than the MTTF when \( \alpha = 0 \) (independence case). It is obvious that the degrees of PQD of FGM copula and Ali-Mikhail-Haq copula are decreasing in \( \alpha \), i.e. \( C(\alpha_i) \geq C(\alpha) \) for \( \alpha_i \leq \alpha, \ i = 1,2 \). From Tables 1, we can see that the more dependence of copula, the smaller MTTF value. The MTTF (without standby component) under FGM model and Ali-Mikhail-Haq model is always a lower bound for a model with a cold standby. And the MTTF in Case 2 is smaller than Case 1.

4 Conclusion

In this paper, we have studied a parallel system with a single standby unit when the components are dependent. The dependences between the components have been modeled by copulas. We compare systems with dependent components whose random vectors of lifetimes have the same copula (that is when they share the same dependence structure). We have presented explicit expression for the MTTF of the system consisting of two components. The findings of the paper are important and novel from various perspectives. This is the first study which considers the effect of components’ dependence through a copula function in the context of a parallel system with a standby unit. In many practical situations, the components in a system are dependent because they share the same environment and this dependence should be considered in system design and management from a practical point of view. An engineer may want know how the lifetime of system is affected by the dependence among the components.

The results of the paper with some modifications can be extended to a parallel system with n components. In this case, the lifetime of the system can be represented as

\[
T = \min(X_1,\ldots,X_n) + \max(X_1^*,\ldots,X_n^*,Z),
\]

where

\[
X_i^* = \min(X_1,\ldots,X_n) \mid X_i > \min(X_1,\ldots,X_n).
\]

The cdf of \( X_i^* \) is

\[
P\{X_i^* \leq t\} = \frac{1}{n-1}[nP\{X_i \leq X_2 + t,\ldots,X_i \leq X_n + t\} - 1].
\]

Similar results can be obtained by choosing an \( n \)-variant copula function \( C(u_1,\ldots,u_n) \).

References


