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Curve integral with path independent in orthogonal curvilinear coordinate system *

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Abstract: It is explored that the line integral is a path independent in two or three arbitrary dimensional orthogonal curvilinear coordinate systems, which is based on the integral condition with the path independent in two or three dimensional rectangular coordinate systems. Firstly, according to the coordinate transformation, the condition that the line integral is the path independent in the polar coordinate system is obtained easily from the Green's theorem in two-dimensional rectangular coordinate system and the condition is extended to arbitrary two-dimension orthogonal curvilinear coordinates. Secondly, through the coordinate transformation relationship and the area projection method, the Stokes formula in three-dimensional rectangular coordinate system is promoted to the spherical coordinate system and cylindrical coordinate system, and the condition that the line integral is a path independent is obtained. Furthermore, the condition is extended to arbitrary three-dimension orthogonal curvilinear coordinates. Lastly, the conclusions are made.

Keywords: curve integral; orthogonal curvilinear coordinate system; coordinate transformation; green's theorem; stokes formula

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1 Introduction

It is well known that the calculation of integral will be much easier if there is no relation between the integral results and the integral path [1-2]. It is of great importance to discuss the decision criteria by which to judge the integral results independent on the path. The independent condition between integral and path can be derived by Green's Theorem in two-dimension rectangular coordinate, and that condition can be got in three-dimension case by Stokes formula [3]. However, it will be in trouble if the integral was dealt with in other coordinate systems, such as in plane polar coordinate, or in cylindrical coordinate, or in spherical coordinate. Therefore, it is significant to derive the independent conditions of the integral results with the integral path in those coordinate systems. Furthermore, if the independent conditions of the integral results with the integral path can be derived in two-dimension arbitrary orthogonal curvilinear coordinate system or three-dimension arbitrary orthogonal coordinate system, it

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will be more significant.

In this paper, the independent condition between integral and integral path (i.e. Green theorem) is used as the starting point in the rectangular coordinate system. Firstly, the condition of the integration independent to the integral path in the plane polar coordinate system is investigated. And the independent condition is extended to two-dimension arbitrary orthogonal curvilinear coordinate systems. Then, based on Stokes formula in the three-dimension rectangular coordinate system, the independent condition of integral with integral path is extended to the cylindrical coordinate system and to the spherical coordinate system. At last, the independent condition is extended to three-dimension arbitrary orthogonal curvilinear coordinate systems.

2 Green's theorem in polar coordinate system and its extension

The Green's theorem in Cartesian coordinate system is

$$\iint_D \frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \mathrm{d}x\,\mathrm{d}y = \oint_l P\,\mathrm{d}x + Q\,\mathrm{d}y \,. \tag{1}$$

According to the theorem, in a single connected region in rectangular coordinate system, if the integral $\oint_l P\,\mathrm{d}x + Q\,\mathrm{d}y$ satisfies $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, the integral along a curve in the area is only related to the start position and the end position, but is not to the integral path. In the calculation of some of the more complex curve points, one can choose a special curve to simplify the calculation. In the calculation of some complex integral along a curve, as long as the start position and end position of the integral keep unchanged, one can choose a special curve to simplify the calculation. But in the actual calculation, the integral along a curve not only is encountered in the Cartesian coordinate system, but also may be encountered along a curve in the other coordinate systems. In this case, one can find out the independent conditions of the integral and the integral path in other coordinate system by doing some simple transformation.

2.1 Green's theorem in polar coordinate

Closed curve integral in the rectangular coordinate can be translated to the plane polar coordinate.

$$\oint_l P\,\mathrm{d}x + Q\,\mathrm{d}y = \oint_l p\,\mathrm{d}r + rq\,\mathrm{d}\theta \,, \tag{2}$$

where

$$\begin{cases} p = P\cos\theta + Q\sin\theta, \\ q = Q\cos\theta - P\sin\theta, \end{cases} \tag{3}$$

and

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \mathrm{d}x\,\mathrm{d}y = \iint_D r\frac{\partial q}{\partial r} - \frac{\partial p}{\partial \theta} \mathrm{d}r\,\mathrm{d}\theta \,. \tag{4}$$

According to Green's theorem,

$$\iint_D \frac{\partial q}{\partial r} - \frac{1}{r}\frac{\partial p}{\partial \theta} \mathrm{d}r\,\mathrm{d}\theta = \oint_l p\,\mathrm{d}r + rq\,\mathrm{d}\theta \,. \tag{5}$$

Eq. (5) is Green's theorem in polar coordinate system. In the polar coordinate system in a single connected region, if there are two known functions which have a continuous first order partial derivative, and satisfy the condition $\frac{\partial q}{\partial r} = \frac{1}{r}\frac{\partial p}{\partial \theta}$, the integral along any path which the starting point and end point keep unchanged is the same, namely, the integral is not related to the integral path.

2.2 Green theorem in arbitrary two-dimension orthogonal curvilinear coordinate system

In fact, for arbitrary two-dimensional orthogonal curvilinear coordinates system (x', y'), one can get the relationship between the two-dimension orthogonal curvilinear coordinates system with the two-dimension Cartesian coordinate system through the coordinate transformation.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}, \tag{6}$$

where

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{pmatrix}. \tag{7}$$

In this coordinate system, the curve integral can be discussed as

$$\oint_l P \,\mathrm{d}\, x + Q \,\mathrm{d}\, y = \oint_l P(g_{11} + g_{21}) \,\mathrm{d}\, x' + Q(g_{12} + g_{22}) \,\mathrm{d}\, y', \tag{8}$$

and

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \,\mathrm{d}\, x \,\mathrm{d}\, y = \iint_D (g'_{11} \frac{\partial Q}{\partial x'} - g'_{21} \frac{\partial P}{\partial x'} + g'_{12} \frac{\partial Q}{\partial y'} - g'_{22} \frac{\partial P}{\partial y'}) \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \,\mathrm{d}\, x' \,\mathrm{d}\, y', \tag{9}$$

in which $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ and $\begin{pmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{pmatrix}$ are inverse matrices. Based on the Green's theorem, there is

$$\iint_D (g'_{11} \frac{\partial Q}{\partial x'} - g'_{21} \frac{\partial P}{\partial x'} + g'_{12} \frac{\partial Q}{\partial y'} - g'_{22} \frac{\partial P}{\partial y'}) \begin{vmatrix} g_{11} & g_{11} \\ g_{11} & g_{11} \end{vmatrix} \,\mathrm{d}\, x' \,\mathrm{d}\, y' = \oint_l P(g_{11} + g_{21}) \,\mathrm{d}\, x' + Q(g_{12} + g_{22}) \,\mathrm{d}\, y'. \tag{10}$$

Eq. (8) is the Green theorem in arbitrary two-dimension orthogonal curvilinear coordinates system. If only

$$(g'_{11} \frac{\partial Q}{\partial x'} - g'_{21} \frac{\partial P}{\partial x'} + g'_{12} \frac{\partial Q}{\partial y'} - g'_{22} \frac{\partial P}{\partial y'}) \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = 0, \tag{11}$$

the integral along any path which the starting point and end point keep unchanged is the same in that coordinates system. Namely, the integral $\int_l P(g_{11} + g_{21}) \,\mathrm{d}\, x' + Q(g_{12} + g_{22}) \,\mathrm{d}\, y'$ is not related to the integral path.

3 Stokes formulas in cylindrical coordinate system and spherical coordinate system and their extension

Based on above discussions, it is obvious that the closed curvilinear integral in 3-dimension rectangular coordinate system can also be transformed into the integral in cylindrical coordinate system and spherical coordinate system.

3.1 Stokes formula in cylindrical coordinate system

The Stokes formula in Cartesian coordinate system is [1,4-6]

$$\iint_\Sigma (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}) \,\mathrm{d}\, y \,\mathrm{d}\, z + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}) \,\mathrm{d}\, z \,\mathrm{d}\, x + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \,\mathrm{d}\, x \,\mathrm{d}\, y = \oint_L P \,\mathrm{d}\, x + Q \,\mathrm{d}\, y + R \,\mathrm{d}\, z, \tag{12}$$

where L is the direction smooth closed curve in three-dimension Cartesian coordinate system, and Σ is a Directed curve surface with closed curve L as its boundary. If the integral $\int_L P \,\mathrm{d}\, x + Q \,\mathrm{d}\, y + R \,\mathrm{d}\, z$ satisfies

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \text{ and } \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \tag{13}$$

the integral along an arbitrary curve in the area is only related to the start position and end position, but not related to the integral path. In the calculation of some of the more complex curve integrals, one can choose a special curve to simplify the calculation if the integrands satisfy Eq. (12).

According to the coordinate transformation in three-dimension cylindrical coordinate system with three-dimension Cartesian coordinate system, the closed curvilinear integral can be written as

$$\oint_L F_1\,\mathrm{d}x+F_2\,\mathrm{d}y+F_3\,\mathrm{d}z=\oint_L (F_1\cos\theta+F_2\sin\theta)\,\mathrm{d}r+ (F_2\cos\theta-F_1\sin\theta)r\,\mathrm{d}\theta+F_3\,\mathrm{d}z\,, \tag{14}$$

and

$$\iint_\Sigma (\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y})\,\mathrm{d}x\,\mathrm{d}y+(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z})\,\mathrm{d}z\,\mathrm{d}y+ (\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x})\,\mathrm{d}x\,\mathrm{d}z=\iint_\Sigma [\frac{\partial(F_2\cos\theta-F_1\sin\theta)}{\partial r}- \frac{\sin\theta}{r}\frac{\partial F_2}{\partial\theta}-\frac{\cos\theta}{r}\frac{\partial F_1}{\partial\theta}]r\,\mathrm{d}r\,\mathrm{d}\theta+[r\sin 2\theta\frac{\partial F_3}{\partial r}+ \cos 2\theta\frac{\partial F_3}{\partial\theta}-r\frac{\partial(F_2\cos\theta+F_1\sin\theta)}{\partial z}]\,\mathrm{d}z\,\mathrm{d}\theta+ [\frac{\partial(F_1\cos\theta-F_2\sin\theta)}{\partial z}-\frac{\partial F_3}{\partial r}+\frac{\sin 2\theta}{r}\frac{\partial F_3}{\partial\theta}]\,\mathrm{d}r\,\mathrm{d}z. \tag{15}$$

Compared to Stokes formula in three-dimension Cartesian coordinate system, with a view to Eqs. (14) and (15), it can be obtained that

$$\oint_L (F_1\cos\theta+F_2\sin\theta)\,\mathrm{d}r+(F_2\cos\theta-F_1\sin\theta)r\,\mathrm{d}\theta+ F_3\,\mathrm{d}z=\iint_\Sigma [\frac{\partial(F_2\cos\theta-F_1\sin\theta)}{\partial r}-\frac{\sin\theta}{r}\frac{\partial F_2}{\partial\theta}- \frac{\cos\theta}{r}\frac{\partial F_1}{\partial\theta}]r\,\mathrm{d}r\,\mathrm{d}\theta+[r\sin 2\theta\frac{\partial F_3}{\partial r}+\cos 2\theta\frac{\partial F_3}{\partial\theta}- r\frac{\partial(F_2\cos\theta+F_1\sin\theta)}{\partial z}]\,\mathrm{d}z\,\mathrm{d}\theta+ [\frac{\partial(F_1\cos\theta-F_2\sin\theta)}{\partial z}-\frac{\partial F_3}{\partial r}+ \frac{\sin 2\theta}{r}\frac{\partial F_3}{\partial\theta}]\,\mathrm{d}r\,\mathrm{d}z\,. \tag{16}$$

Eq. (15) is the Stokes formula in cylindrical coordinate. If integrand functions satisfy

$$\begin{cases}\cos\theta\frac{\partial F_2}{\partial r}=\sin\theta\frac{\partial F_1}{\partial r}+\frac{\sin\theta}{r}\frac{\partial F_2}{\partial\theta}+\frac{\cos\theta}{r}\frac{\partial F_1}{\partial\theta}, \\ r\sin 2\theta\frac{\partial F_3}{\partial r}+\cos 2\theta\frac{\partial F_3}{\partial\theta}=r\cos\theta\frac{\partial F_2}{\partial z}+r\sin\theta\frac{\partial F_1}{\partial z}, \\ \cos\theta\frac{\partial F_1}{\partial z}+\frac{\sin 2\theta}{r}\frac{\partial F_3}{\partial\theta}=\frac{\partial F_3}{\partial r}+\sin\theta\frac{\partial F_2}{\partial z},\end{cases} \tag{17}$$

then the curve integral will be independent to the integral path.

3.2 Stokes formula in spherical coordinate system

Similarly, according to the coordinate transformation in spherical coordinate system with three-dimension Cartesian coordinate system, the closed curvilinear integral can be written as

$$\oint_L F_1\,\mathrm{d}x+F_2\,\mathrm{d}y+F_3\,\mathrm{d}z= \oint_L (F_1\sin\varphi\cos\theta+F_2\sin\varphi\sin\theta+F_3\cos\varphi)\,\mathrm{d}r+ r\sin\varphi(F_2\cos\theta-F_1\sin\theta)\,\mathrm{d}\theta+r(F_1\cos\varphi\cos\theta+ F_2\cos\varphi\sin\theta-F_3\sin\varphi)\,\mathrm{d}\varphi\,, \tag{18}$$

and

$$\iint_\Sigma \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\,\mathrm{d}x\,\mathrm{d}y+\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\,\mathrm{d}z\,\mathrm{d}y+\frac{\partial F_1}{\partial z}- \frac{\partial F_3}{\partial x}\,\mathrm{d}x\,\mathrm{d}z=\iint_\Sigma [r\sin\varphi\cos\theta\frac{\partial F_2}{\partial r}-\sin\theta\sin^2\varphi\frac{\partial F_2}{\partial\theta}+ 2\cos\theta\cos\varphi\sin^2\varphi\frac{\partial F_2}{\partial\varphi}-r\sin\varphi\sin\theta\frac{\partial F_1}{\partial r}- \cos\theta\sin^2\varphi\frac{\partial F_1}{\partial\theta}-2\sin\theta\cos\varphi\sin^2\varphi\frac{\partial F_1}{\partial\varphi}- \cos\varphi\sin\varphi\frac{\partial F_3}{\partial\theta}]\,\mathrm{d}r\,\mathrm{d}\theta+[r\sin\varphi\cos\varphi\sin\theta\frac{\partial F_2}{\partial\theta}+ (\sin^2\varphi-\cos^2\varphi)r\sin\varphi\cos\theta\frac{\partial F_2}{\partial\theta}+R\sin\varphi\cos\varphi\frac{\partial F_1}{\partial\theta}+ \cos^2\varphi-\sin^2\varphi)r\sin\varphi\sin\theta\frac{\partial F_1}{\partial\varphi}- r\sin^2\frac{\partial F_3}{\partial\theta}]\,\mathrm{d}\theta\,\mathrm{d}\varphi+[r\sin\varphi\frac{\partial F_3}{\partial r}+\cos\varphi\frac{\partial F_3}{\partial\varphi}-$$

$$r\sin\theta\cos\varphi\frac{\partial F_2}{\partial r}-\sin\theta\sin\varphi\frac{\partial F_2}{\partial\varphi}-$$
$$r\cos\theta\cos\varphi\frac{\partial F_1}{\partial r}-\cos\theta\sin\varphi\frac{\partial F_1}{\partial\varphi}]\mathrm{d}\varphi\,\mathrm{d}r\,. \qquad (19)$$

Compared to Stokes formula in three-dimension Cartesian coordinate system, with a view to Eqs. (18) and (19), Stokes formula can be gotten in sphere-coordinate system as

$$\oint_L(F_1\sin\varphi\cos\theta+F_2\sin\varphi\sin\theta+F_3\cos\varphi)\,\mathrm{d}r+$$
$$r\sin\varphi(F_2\cos\theta-F_1\sin\theta)\,\mathrm{d}\theta+$$
$$r(F_1\cos\varphi\cos\theta+F_2\cos\varphi\sin\theta-F_3\sin\varphi)\,\mathrm{d}\varphi=$$
$$\iint_\Sigma[r\sin\varphi\cos\theta\frac{\partial F_2}{\partial r}-\sin\theta\sin^2\varphi\frac{\partial F_2}{\partial\theta}+$$
$$2\cos\theta\cos\varphi\sin^2\varphi\frac{\partial F_2}{\partial\varphi}-r\sin\varphi\sin\theta\frac{\partial F_1}{\partial r}-$$
$$\cos\theta\sin^2\varphi\frac{\partial F_1}{\partial\theta}-2\sin\theta\cos\varphi\sin^2\varphi\frac{\partial F_1}{\partial\varphi}-$$
$$\cos\varphi\sin\varphi\frac{\partial F_3}{\partial\theta}]\mathrm{d}r\,\mathrm{d}\theta+[r\sin\varphi\cos\varphi\sin\theta\frac{\partial F_2}{\partial\theta}+$$
$$(\sin^2\varphi-\cos^2\varphi)r\sin\varphi\cos\theta\frac{\partial F_2}{\partial\theta}+R\sin\varphi\cos\varphi\frac{\partial F_1}{\partial\theta}+$$
$$\cos^2\varphi-\sin^2\varphi)r\sin\varphi\sin\theta\frac{\partial F_1}{\partial\varphi}-r\sin^2\frac{\partial F_3}{\partial\theta}]\mathrm{d}\theta\,\mathrm{d}\varphi+$$
$$[r\sin\varphi\frac{\partial F_3}{\partial r}+\cos\varphi\frac{\partial F_3}{\partial\varphi}-r\sin\theta\cos\varphi\frac{\partial F_2}{\partial r}-$$
$$\sin\theta\sin\varphi\frac{\partial F_2}{\partial\varphi}-r\cos\theta\cos\varphi\frac{\partial F_1}{\partial r}-$$
$$\cos\theta\sin\varphi\frac{\partial F_1}{\partial\varphi}]\mathrm{d}\varphi\,\mathrm{d}r\,. \qquad (20)$$

If integrand functions satisfy

$$\begin{cases}F_1\sin\varphi\cos\theta+F_2\sin\varphi\sin\theta+F_3\cos\varphi=0,\\ r(F_1\cos\varphi\cos\theta+F_2\cos\varphi\sin\theta-F_3\sin\varphi)=0,\\ r\sin\varphi(F_2\cos\theta-F_1\sin\theta)=0,\end{cases} \qquad (21)$$

then the curve integral will be independent to the integral path.

3.3 Stokes formula in orthogonal curvilinear coordinates

Furthermore, for arbitrary three-dimensional orthogonal curvilinear coordinates system (x',y',z'), according to the coordinate transformation one can get the relation between arbitrary three-dimensional orthogonal curvilinear coordinates system with the three-dimension Cartesian coordinate system. Based on the coordinate transformation [5,7],

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{\partial x}{\partial x'}&\frac{\partial x}{\partial y'}&\frac{\partial x}{\partial z'}\\ \frac{\partial y}{\partial x}&\frac{\partial y}{\partial y'}&\frac{\partial y}{\partial z'}\\ \frac{\partial z}{\partial x}&\frac{\partial z}{\partial y'}&\frac{\partial z}{\partial z'}\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}\underline{\underline{\mathrm{def}}}\begin{pmatrix}g_{11}&g_{12}&g_{13}\\g_{21}&g_{22}&g_{23}\\g_{31}&g_{32}&g_{33}\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}. \qquad (22)$$

The closed curve integral in the three-dimension (x',y',z') can be changed into

$$\oint_L P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z=\oint_L(Pg_{11}+Qg_{21}+Rg_{31})\,\mathrm{d}x'+$$
$$(Pg_{12}+Qg_{22}+Rg_{32})\,\mathrm{d}y'+$$
$$(Pg_{13}+Qg_{23}+Rg_{33})\,\mathrm{d}z', \qquad (23)$$

where *P*, *Q* and *R* are three functions, and

$$\oint_L P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z=\iint_\Sigma(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})\,\mathrm{d}x\,\mathrm{d}y+$$
$$(\frac{\partial P}{\partial y}-\frac{\partial R}{\partial z})\,\mathrm{d}y\,\mathrm{d}z+(\frac{\partial R}{\partial z}-\frac{\partial Q}{\partial x})\,\mathrm{d}x\,\mathrm{d}z\,. \qquad (24)$$

Because

$$\mathrm{d}x\,\mathrm{d}y=\begin{vmatrix}g_{11}&g_{12}\\g_{21}&g_{22}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}y'-\begin{vmatrix}g_{11}&g_{13}\\g_{21}&g_{23}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}z'+$$
$$\begin{vmatrix}g_{12}&g_{13}\\g_{22}&g_{23}\end{vmatrix}\mathrm{d}y'\,\mathrm{d}z', \qquad (25)$$

$$\mathrm{d}x\,\mathrm{d}z=\begin{vmatrix} g_{12} & g_{11} \\ g_{32} & g_{31}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}y'+\begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}z'-\begin{vmatrix} g_{12} & g_{13} \\ g_{32} & g_{33}\end{vmatrix}\mathrm{d}y'\,\mathrm{d}z', \tag{26}$$

and

$$\mathrm{d}y\,\mathrm{d}z=\begin{vmatrix} g_{21} & g_{22} \\ g_{31} & g_{32}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}y'-\begin{vmatrix} g_{21} & g_{23} \\ g_{31} & g_{33}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}z'+\begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33}\end{vmatrix}\mathrm{d}y'\,\mathrm{d}z', \tag{27}$$

the integral will be turned into

$$\iint_{\Sigma}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y+\left(\frac{\partial P}{\partial y}-\frac{\partial R}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z+\left(\frac{\partial R}{\partial z}-\frac{\partial Q}{\partial x}\right)\mathrm{d}x\,\mathrm{d}z=\iint_{\Sigma}\left(\frac{\partial Q}{\partial x'}g'_{11}+\frac{\partial Q}{\partial y'}g'_{12}+\frac{\partial Q}{\partial z'}g'_{13}-\frac{\partial P}{\partial x'}g'_{21}-\frac{\partial P}{\partial y'}g'_{22}-\frac{\partial P}{\partial z'}g'_{23}\right)\times\left(\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}y'-\begin{vmatrix} g_{11} & g_{13} \\ g_{21} & g_{23}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}z'+\begin{vmatrix} g_{12} & g_{13} \\ g_{22} & g_{23}\end{vmatrix}\mathrm{d}y'\,\mathrm{d}z'\right)+\left(\frac{\partial P}{\partial x'}g'_{21}+\frac{\partial P}{\partial y'}g'_{22}+\frac{\partial P}{\partial z'}g'_{23}-\frac{\partial R}{\partial x'}g'_{31}-\frac{\partial R}{\partial y'}g'_{32}-\frac{\partial R}{\partial z'}g'_{33}\right)\times\left(\begin{vmatrix} g_{21} & g_{22} \\ g_{31} & g_{32}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}y'-\begin{vmatrix} g_{21} & g_{23} \\ g_{31} & g_{33}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}z'+\begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33}\end{vmatrix}\mathrm{d}y'\,\mathrm{d}z'\right)+\left(\frac{\partial R}{\partial x'}g'_{11}+\frac{\partial R}{\partial y'}g'_{12}+\frac{\partial R}{\partial z'}g'_{13}-\frac{\partial Q}{\partial x'}g'_{31}-\frac{\partial Q}{\partial y'}g'_{32}-\frac{\partial Q}{\partial z'}g'_{33}\right)\times\left(\begin{vmatrix} g_{12} & g_{11} \\ g_{32} & g_{31}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}y'+\begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}z'-\begin{vmatrix} g_{12} & g_{13} \\ g_{32} & g_{33}\end{vmatrix}\mathrm{d}y'\,\mathrm{d}z'\right). \tag{28}$$

where $\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33}\end{pmatrix}$ and $\begin{pmatrix} g'_{11} & g'_{12} & g'_{13} \\ g'_{21} & g'_{22} & g'_{23} \\ g'_{31} & g'_{32} & g'_{33}\end{pmatrix}$ are reciprocal, and Stokes formula is

$$\oint_{L}(Pg_{11}+Qg_{21}+Rg_{31})\,\mathrm{d}x'+(Pg_{12}+Qg_{22}+Rg_{32})\,\mathrm{d}x'+(Pg_{13}+Qg_{23}+Rg_{33})\,\mathrm{d}x'=\iint_{\Sigma}\left(\frac{\partial Q}{\partial x'}g'_{11}+\frac{\partial Q}{\partial y'}g'_{12}+\frac{\partial Q}{\partial z'}g'_{13}-\frac{\partial P}{\partial x'}g'_{21}-\frac{\partial P}{\partial y'}g'_{22}-\frac{\partial P}{\partial z'}g'_{23}\right)\times\left(\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}y'-\begin{vmatrix} g_{11} & g_{13} \\ g_{21} & g_{23}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}z'+\begin{vmatrix} g_{12} & g_{13} \\ g_{22} & g_{23}\end{vmatrix}\mathrm{d}y'\,\mathrm{d}z'\right)+\left(\frac{\partial P}{\partial x'}g'_{21}+\frac{\partial P}{\partial y'}g'_{22}+\frac{\partial P}{\partial z'}g'_{23}-\frac{\partial R}{\partial x'}g'_{31}-\frac{\partial R}{\partial y'}g'_{32}-\frac{\partial R}{\partial z'}g'_{33}\right)\times\left(\begin{vmatrix} g_{21} & g_{22} \\ g_{31} & g_{32}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}y'-\begin{vmatrix} g_{21} & g_{23} \\ g_{31} & g_{33}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}z'+\begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33}\end{vmatrix}\mathrm{d}y'\,\mathrm{d}z'\right)+\left(\frac{\partial R}{\partial x'}g'_{11}+\frac{\partial R}{\partial y'}g'_{12}+\frac{\partial R}{\partial z'}g'_{13}-\frac{\partial Q}{\partial x'}g'_{31}-\frac{\partial Q}{\partial y'}g'_{32}-\frac{\partial Q}{\partial z'}g'_{33}\right)\times\left(\begin{vmatrix} g_{12} & g_{11} \\ g_{32} & g_{31}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}y'+\begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33}\end{vmatrix}\mathrm{d}x'\,\mathrm{d}z'-\begin{vmatrix} g_{12} & g_{13} \\ g_{32} & g_{33}\end{vmatrix}\mathrm{d}y'\,\mathrm{d}z'\right). \tag{29}$$

Eq. (27) is the Green formula for the integral under the coordinate system of any orthogonal curvilinear coordinates.

If integrand functions satisfy

$$\left(\frac{\partial Q}{\partial x'}g'_{11}+\frac{\partial Q}{\partial y'}g'_{12}+\frac{\partial Q}{\partial z'}g'_{13}-\frac{\partial P}{\partial x'}g'_{21}-\frac{\partial P}{\partial y'}g'_{22}-\frac{\partial P}{\partial z'}g'_{23}\right)\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{vmatrix}+\left(\frac{\partial P}{\partial x'}g'_{21}+\frac{\partial P}{\partial y'}g'_{22}+\frac{\partial P}{\partial z'}g'_{23}-\frac{\partial R}{\partial x'}g'_{31}-\frac{\partial R}{\partial y'}g'_{32}-\frac{\partial R}{\partial z'}g'_{33}\right)\begin{vmatrix} g_{21} & g_{22} \\ g_{31} & g_{32}\end{vmatrix}+\left(\frac{\partial R}{\partial x'}g'_{11}+\frac{\partial R}{\partial y'}g'_{12}+\frac{\partial R}{\partial z'}g'_{13}-\frac{\partial Q}{\partial x'}g'_{31}-\frac{\partial Q}{\partial y'}g'_{32}-\frac{\partial Q}{\partial z'}g'_{33}\right)\begin{vmatrix} g_{12} & g_{11} \\ g_{32} & g_{31}\end{vmatrix}=0, \tag{30}$$

$$\left(\frac{\partial R}{\partial x'}g'_{11}+\frac{\partial R}{\partial y'}g'_{12}+\frac{\partial R}{\partial z'}g'_{13}-\frac{\partial Q}{\partial x'}g'_{31}-\frac{\partial Q}{\partial y'}g'_{32}-\right.$$

$$\frac{\partial Q}{\partial z'}g'_{33})\begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{vmatrix} - (\frac{\partial P}{\partial x'}g'_{21} + \frac{\partial P}{\partial y'}g'_{22} + \frac{\partial P}{\partial z'}g'_{23} - \frac{\partial R}{\partial x'}g'_{31} - \frac{\partial R}{\partial y'}g'_{32} - \frac{\partial R}{\partial z'}g'_{33})\begin{vmatrix} g_{21} & g_{23} \\ g_{31} & g_{33} \end{vmatrix} - (\frac{\partial Q}{\partial x'}g'_{11} + \frac{\partial Q}{\partial y'}g'_{12} + \frac{\partial Q}{\partial z'}g'_{13} - \frac{\partial P}{\partial x'}g'_{21} - \frac{\partial P}{\partial y'}g'_{22} - \frac{\partial P}{\partial z'}g'_{23})\begin{vmatrix} g_{11} & g_{13} \\ g_{21} & g_{23} \end{vmatrix} = 0, \quad (31)$$

and

$$(\frac{\partial Q}{\partial x'}g'_{11} + \frac{\partial Q}{\partial y'}g'_{12} + \frac{\partial Q}{\partial z'}g'_{13} - \frac{\partial P}{\partial x'}g'_{21} - \frac{\partial P}{\partial y'}g'_{22} - \frac{\partial P}{\partial z'}g'_{23})\begin{vmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{vmatrix} + (\frac{\partial P}{\partial x'}g'_{21} + \frac{\partial P}{\partial y'}g'_{22} + \frac{\partial P}{\partial z'}g'_{23} - \frac{\partial R}{\partial x'}g'_{31} - \frac{\partial R}{\partial y'}g'_{32} - \frac{\partial R}{\partial z'}g'_{33})\begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} - (\frac{\partial R}{\partial x'}g'_{11} + \frac{\partial R}{\partial y'}g'_{12} + \frac{\partial R}{\partial z'}g'_{13} - \frac{\partial Q}{\partial x'}g'_{31} - \frac{\partial Q}{\partial y'}g'_{32} - \frac{\partial Q}{\partial z'}g'_{33})\begin{vmatrix} g_{12} & g_{13} \\ g_{32} & g_{33} \end{vmatrix} = 0. \quad (32)$$

Then the curve integral will be independent to the integral path.

4 Discussion

In short, the condition is explored that the line integral is path independent in two or three curvilinear coordinate systems, even in two or three dimensional arbitrary orthogonal curvilinear coordinate systems. But the conditions of the integral with path independent become more complex with the increasing of dimensions of coordinate systems. And if the integral is discussed in *n*-dimension arbitrary orthogonal curvilinear coordinate systems, the conditions of the integral with path independent will become more complex. And that will be the content of the next step of our research.

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