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Infinitely Many Non-collision Periodic Solutions for N-Body-Type Problems*

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ABSTRACT We prove the existence of infinitely many non-collision T-periodic solutions for N-body-type problems: $-\ddot{u}_i = V_{u_i}(u,t)$. Where $u=(u_1,\cdots,u_N)$, $u_i \in R^k$ and $V(u,t)=\sum_{1\leqslant i\neq j\leqslant N} V_{ij}(u_i - u_j, t)$, the potentials $V_{ij}(\xi)$ are T-periodic in t and singular at $\xi=0$ which satisfy the strong force condition of Gordon.

The proofs are based on a variant of the perturbation methods of K. Uhlenbeck, W. Ding, P. Majer and S. Terracini.

KEYWORDS N- body- type problems; infinitely many non- collision periodic solutions; Ljusternik-Schnirelmann theory

0 Introduction and Main Results

Concerning the n- body- type problems, symmetrical cases have been studied in [1 ~ 6]. Concerning the three-body- type problem, non- symmetrical cases have been treated by Bahri and Rabinowitz in [7].

In [8], P. Majer presented two variational methods on manifolds with boundary. This theory is essentially a perturbation argument of avoiding the difficulties of a lack of the Palais- Smale condition or other compactness conditions of the variational functional. Majer and Terracini[9~11], Ambrosetti, Tanaka and Vitillaro[12] applied the argument to the study of the periodic solutions of Hamiltonian systems and obtained good results. The perturbation method for harmonic maps and other maps from which the critical points of some energy integrals may be found out was proposed by K. Uhlenbeck[13] and W. Ding[14]. In this paper, we shall deal with the N-body-type problems for non-symmetrical potentials using a variant of the perturbation method of [8~11,13~14].

We look for periodic solutions of N-body-type problems of the type: $-\ddot{u}_i = V_{u_i}(u_T,t)$ where $V_{ij} \in C^1((R^k\backslash\{0\})\times R, R)$ are T-periodic in t. We shall consider the following assumptions on the potential $V_{ij}(i,j=1,\cdots,N, i\neq j)$:

(V1) $V_{ij}(x,t)=V_{ji}(-x,t), \forall x \in R^k\backslash\{0\}$;

(V2) $V_{ij}(x,t)\leqslant 0, \forall x\in R^k\backslash\{0\}, \forall t \in R$;

(V3) V_{ij} satisfies the strong force condition of Gordon ([15]), that is, there are $\rho_0>0$, $U\in ((R^k\backslash\{0\}; R)$ such that

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$\lim_{x\to 0} U(x)=+\infty, -V_{ij}(x,t)\geqslant |U'(x)|^2$, for $0<|x|<\rho_0$.

(V4) There are $\rho>0$ and $0\leqslant\theta<\frac{\pi}{2}$ such that $\text{ang}(V'_{ij}(x,t),x)\geqslant\pi-\theta, \forall\ x, |x|>\rho$

Where $V'_{ij}(x,t)$ denote $\left(\frac{\partial}{\partial x_1}V_{ij}(x,t),\cdots,\frac{\partial}{\partial x_k}V_{ij}(x,t)\right)\in R^k$ and in any euclidean space $0\leqslant$ ang $(x,y)\leqslant\pi$ denotes the angle between x and y.

We will say that a function $u(t)=(u_1(t),\cdots,u_N(t))\in C^2(R,(R^k)^N\setminus\{0\})$ is a T-periodic non-collision solution of $-\ddot{u}_i=V'_{u_i}(u,t)$ if u is a T-periodic solution of $-\ddot{u}_i=V'_{u_i}(u,t)$ and $u_i(t)\neq u_j(t)$ for all $i\neq j$ and $t\in R$.

Our main results are the following:

Theorem 1. Assume (V1) ~ (V4) hold, then $-\ddot{u}_i=V'_{u_i}(u,t)$ has infinitely many non-collision T-periodic solutions.

1 Ljusternik-Schnirelmann Type Theorem with Perturbation Palais-Smale Conditions

Motivated by the work [14] of Ding and the work [8] of Majer we have

Theorem 2. Let X be a Banach space and Λ be an open subset of X, let $f\in C^1(\Lambda;R)$. We pose the following assumptions on f and g.

(A0) $\lim_{v\to+\infty} f(x_v)=+\infty$, if $x_v\to x_0\in\partial\Lambda$,

(A1) f is bounded from below; $g\geqslant 0$ and $\|g'(x)\|$ is bounded on any set where $g(x)$ is bounded.

(A2) For $\lambda>0$, $f+\lambda g$ satisfies (PS) condition.

(A3) If $\{u_i\}\subset\Lambda$ is a sequence such that $f(u_i)\to c, f'(u_i)\to 0$ and $g(u_i)$ is bounded, then c is a critical value of f.

(A4) There exist $\bar{\lambda}>0$ and $\bar{c}\leqslant\infty$ with the property that for any $c<\bar{c}$ there exists $\beta=\beta(c)$ such that if u is a critical point of $f+\lambda g$ with $\lambda\in[0,\bar{\lambda}]$ and $f(u)+\lambda g(u)\leqslant c$, then $g(u)\leqslant\beta$. Let

$$c_k=c_k(f)=\inf\{\sup f(u)\,|\,\text{cat}_\Lambda(A)\geqslant k\},\quad 1\leqslant k\leqslant\text{cat}_\Lambda(\Lambda)$$

If $c_k<\bar{c}$, where $\bar{c}$ is given by (A4), then c_k is critical value of f. If $\bar{c}=\infty$ and $c_k=\infty$ for some k, then the critical values of f are unbounded.

Proof. Note that a lack of completeness about Λ is due to the fact that the domain of f is open; Condition (A0) is a standard way to avoid this difficulty. The rest of the proof of Theorem 2 is similar to that of Ding [14] and Majer [8].

2 The Proofs of Theorem 1.

We introduce spaces:

$$E=\{u=(u_1,\cdots,u_N)\,|\,u_i\in H^1(R/TZ;R^k)\}$$

$$E_0 = \left\{u \in E \,\middle|\, \sum_{i=1}^{N}[u_i] = 0\right\} = E/R^k$$

$$\Lambda = \{u \in E \mid u_i(t) \neq u_j(t), \forall\, t, i \neq j\}$$

$$\Lambda_0 = \Lambda/R^k = \left\{u \in \Lambda \,\middle|\, \sum_{i=1}^{N}[u_i] = 0\right\}$$

Where $H^1(R/TZ; R^k)$ is the Sobolev space of T-periodic function with square summable first derivatives under the norm: $\| u \|_E = \left(\int_0^T |\dot{u}|^2 dt + [u]^2 \right)^{1/2}$

where $[u] = \frac{1}{T}\int_0^T u(t)dt$ denotes the mean value of u.

In order to prove Theorem 1., we shall apply theorem 2. in the following situation:

$$f(u) = \frac{1}{2}\int_0^T |\dot{u}|^2 dt - \int_0^T V(u,t)dt, \qquad \forall\, u \in \Lambda_0$$

$$g(u) = [u]^2 = \frac{1}{2N}\sum_{i,j=1}^{N}\left(\frac{1}{T}\int_0^T (u_i(t) - u_j(t))dt\right)^2$$

It is easy to know that a critical point of f on Λ_0 is a noncollision solution of $-\ddot{u}_i = V'_{x_i}(u,t)$.

Lemma 1.[15] Condition (V3) implies (A0), that is, for every $c>0$, there is $\delta(c)>0$ such that

$$f(u) \leqslant c \Rightarrow \min_{t\in R}|u_i(t) - u_j(t)| \geqslant \delta(c)$$

Lemma 2. Condition (V2) implies (A1).

Proof. Obviously $g(u) \geqslant 0$ and by (V2), we know that $f(u) \geqslant 0$ for any $u \in \Lambda_0$.

Now for any $u, v \in \Lambda_0$, we have

$$|g(u+v) - g(u) - 2[u][v]| =$$

$$\frac{1}{T^2}\left|\left(\int_0^T (u+v)dt\right)^2 - \left(\int_0^T u dt\right)^2 - 2\int_0^T u dt \int_0^T v dt\right| =$$

$$\frac{1}{T^2}\left|\int_0^T v dt\right|^2 \leqslant \frac{1}{T^2}\left[\left(\int_0^T 1 dt\right)^{\frac{1}{2}}\left(\int_0^T |v|^2 dt\right)^{\frac{1}{2}}\right]^2 =$$

$$\frac{1}{T^2}(T \cdot \| v \|_{L^2}^2) = \frac{1}{T}\| v \|_{L^2}^2 \leqslant \frac{c}{T}\| v \|_{E_0}^2$$

So we obtain $(g'(u), v)_{E_0} = 2[u][v], \forall\, u, v \in E_0$

$$\sup_{\|v\|_{E_0}=1}|g'(u), v)_{E_0}| = \sup_{\|v\|_{E_0}=1}|2[u][v]| = 2|[u]| \sup_{\|v\|_{E_0}=1}|[v]| \leqslant 2|[u]|$$

That is $\| g'(u) \|_{E_0} \leqslant 2|[u]|$

Let $\Omega = \{u \in \Lambda_0 \mid g(u) = [u]^2 \leqslant a\}$, then for any $u \in \Omega$,

$$\| g'(u) \|_{E_0} \leqslant 2|[u]| \leqslant 2\sqrt{a}$$

That is $\| g'(u) \|_{E_0}$ is bounded on any set where $g(u)$ is bounded.

Lemma 3. If (V1)～(V3) hold, then (A2) also holds, that is for every $\lambda>0$ and $c \in R$, every sequence $\{u_n\} \subset \Lambda_0$ such that $f(u_n) + \lambda g(u_n) \to c$, $f'(u_n) + \lambda g'(u_n) \to 0$ possesses a converging subsequence.

Proof. For every $\lambda>0$ and $c\in R$, every sequence $\{u_v\}\subset\Lambda_0$ such that $f(u_v)+\lambda g(u_v)\to c$, we know that $\| \dot{u}_v \|_{L^2}$ and $[u_v]^2$ are bounded by (V2). Hence u_v is E_0 — bounded and therefore the existence of a subsequence converging in the weak topology of E_0 and in the uniform topology to some $u\in E_0$, from Lemma 1. it follows that $u\in\Lambda_0$. Hence $\langle V_u'(u_v,t), u-u_v\rangle$ converges uniformly to zero. Since $f'(u_v)+\lambda g'(u_v)\to 0$ and $u-u_v$ is E_0-bounded, and g' is compact, we have

$$\| \dot{u} \|_{L^2}^2 - \lim_{v\to\infty} \| \dot{u}_v \|_{L^2}^2 = \lim_{v\to\infty} \int_0^T \langle \dot{u}_v, \dot{u} - \dot{u}_v\rangle dt =$$

$$\lim_{v\to\infty}\left\{\langle f'(u_v) + \lambda g'(u_v), u - u_v\rangle - \lambda\langle g'(u_v), u - u_v\rangle + \int_0^T \langle V_u(u_v,t), u - u_v\rangle dt\right\} = 0$$

Therefore u_v converges to u strongly in E_0.

Lemma 4. Assume (V1), (V2) and (V3) hold, then (A3) holds.

Remark. The Proof of Lemma 4. is similar to that of Lemma 3..

Lemma 5. (covering Lemma [9]) Let $\rho\geqslant 0$ and $\theta\in\left[0,\frac{\pi}{2}\right)$ be fixed real number. Let $B=\{B(x_i,r)\}_{i\leqslant n}$ be a given family of n balls of R^k. Then (i) there is another family of balls $B'=\{B(x_i',r_i')\}_{i\leqslant n'}$ and a subjective map $\sigma:\{1,\cdots,n\}\to\{1,\cdots,n'\}$ such that, for every $i\leqslant n$, $B(x_i,r)\subseteq B(x'_{\sigma(i)},r'_{\sigma(i)})$; B' verifies:

$$\forall\ i\neq j, \forall\ x\in B(x_i',r_i'), \forall\ y\in B(x_j',r_j')$$

$$\begin{cases} |x-y|>\rho \\ \operatorname{ang}(x-y, x_i'-x_j')<\dfrac{\pi}{2}-\theta \end{cases} \tag{1}$$

and for every $i\leqslant n'$,

$$r_i'\leqslant R(n,\theta)\left(\frac{r}{\cos\theta}+\frac{\rho}{2}\right)+r \tag{2}$$

(ii) If, in addition,

$$\frac{1}{2n}\sum_{i,j=1}^{n}|x_i-x_j|^2>nR(n,\theta)^2\left(\frac{r}{\cos\theta}+\frac{\rho}{2}\right)^2 \tag{3}$$

then $n'\geqslant 2$,

Where

$$R(n,\theta)=\begin{cases} n-1 & \text{if } \theta=0 \\ \dfrac{c(\theta)^{n-1}-1}{c(\theta)-1} & \text{if } \theta>0 \end{cases} \tag{4}$$

and

$$c(\theta)=\frac{1}{2}\left(1+\frac{1}{\cos\theta}\right) \tag{5}$$

Lemma 6. Assume (V1), (V2) and (V3) hold, then for $\bar{\lambda}<\infty$ and $\bar{c}=\infty$ we have that for any $c<\bar{c}$ there exists $\beta=\beta(c)$ such that if $f'(u)+\lambda g'(u)=0$ with $\lambda\in[0,\bar{\lambda}]$ and $f(u)+\lambda g(u)\leqslant c$, then $g(u)\leqslant\beta$, that is (A4) of Theorem 1 holds.

Proof. For any $c<\infty$, let

$$\beta=\beta(c)=NR(N,\theta)^2\left(\sqrt{\frac{cT}{6}}\frac{1}{\cos\theta}+\frac{\rho}{2}\right)^2 \tag{6}$$

To prove Lemma 6., it suffices to prove that for any $\lambda\geqslant 0$ and $u\in\Lambda$ with $f(u)+\lambda g(u)\leqslant c$, $g(u)\leqslant\beta$, there exists a $v\in H^1$ such that

$$\begin{cases} \nabla f(u)\cdot v\geqslant 0 \\ \nabla g(u)\cdot v>0 \end{cases} \tag{7}$$

By (V1), $f(u)+\lambda g(u)\leqslant c$, and Sobolev inequality, we have

$$\| u_i - [u_i] \|_\infty \leqslant \left(\frac{T}{12}\right)^{\frac{1}{2}} (2c)^{\frac{1}{2}} = \left(\frac{Tc}{6}\right)^{\frac{1}{2}} \tag{8}$$

One has $u_i(t) \in B(x_i, r)$ for every $i \leqslant N$ and every $t \in R$, with $r = \left(\frac{Tc}{6}\right)^{\frac{1}{2}}$, and $x_i = [u_i]$. Applying the covering lemma to the family $B = \{B(x_i, r)\}_{i \leqslant N}$, we get the cover $B' = \{B(x_i', r')\}_{i \leqslant N'}$ and the map σ. Moreover $N' \geqslant 2$, since we have supposed $\frac{1}{2N}\sum_{i,j} |x_i - x_j|^2 = g(u) > \beta$

we define, for $i = 1, \cdots, N$ $\qquad v_i = x'_{\sigma(i)} \in \mathrm{R}^k$

then $\nabla f(u) \cdot v = -\frac{1}{2}\sum_{i,j} \int_0^T V_{ij}(u_i - u_j, t) \cdot (x'_{\sigma(i)} - x'_{\sigma(j)}) \mathrm{d}t$

Notice that the only indices (i, j) which contributes to the sum are those for which $\sigma_{(i)} \neq \sigma_{(j)}$; in that case from (8) we have

$$u_i(t) \in B(r_i, r) \text{ and } u_j(t) \in B(x_j, r) \text{ for } t \in \mathrm{R}$$

We get by (2) that, for all $t \in \mathrm{R}$, (denoting for simplicity $u_i = u_i(t), u_j = u_j(t)$)

$$\begin{cases} |u_i - u_j| > \rho \\ \operatorname{ang}(u_i - u_j, x'_{\sigma(i)} - x'_{\sigma(j)}) < \frac{\pi}{2} - \theta \end{cases}$$

From the first of these inequalities and from the hypothesis (V4) on V_{ij} we also have

$$\operatorname{ang}(\nabla V_{ij}(u_i - u_j, t), u_i - u_j) \geqslant \pi - \theta$$

So that ang $(\nabla V_{ij}(u_i - u_j, t), x'_{\sigma(i)} - x'_{\sigma(j)}) > \frac{\pi}{2}$

So we infer $\nabla V_{ij}(u_i - u_j, t), (x'_{\sigma(i)} - x'_{\sigma(j)}) < 0$

Thus $\nabla f(u)\ v > 0$. In a similar way, we have

$$\nabla g(u) \cdot v = 2 \cdot \frac{1}{2N}\sum_{i,j} [u_i - u_j] \cdot (x'_{\sigma(i)} - x'_{\sigma(j)}) > 0$$

for $[u_i - u_j] \cdot (x'_{\sigma(i)} - x'_{\sigma(j)}) > 0$ whenever $\sigma_{(i)} \neq \sigma_{(j)}$ (notice; here enters the fact that $N' \geqslant 2$).

Notice that $\Lambda_0 = \Lambda/\mathrm{R}^k$ and R^k is a finite dimensional Euclidean space, we have

Lemma 7. (Fadell-Husseini[5]) The $\mathrm{Cat}_{\Lambda_0}(\Lambda_0)$ for the Ljusternik-Schnirelmann category of Λ_0 is infinite.

Lemma 8. $C_{+\infty} = C_{+\infty}(f) = \inf\{\sup(u) \mid \mathrm{Cat}_{\Lambda_0}(A) = +\infty\} = +\infty$.

Proof. Obviously, it suffices to prove that for any $\lambda \in \mathrm{R}$, $\mathrm{Cat}_{\Lambda_0}\{u \in \Lambda_0 \mid f(u) \leqslant \lambda\} < +\infty$. The proof of this fact is similar to that of Majer [16].

The Theorem 1 is proved by Lemma 8.

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N 体型问题的无穷多非碰撞周期解

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摘　要　证明了以下的 N 体型问题无穷多个非碰撞 T-周期解的存在性：$-\ddot{u}_i=V_{u_i}(u,t)$. 其中 $u=(u_1,\cdots,u_N)$，$u_i\in R^k$ 且 $V(u,t)=\sum_{1\leq i\neq j\leq N}V_{ij}(u_i-u_j,t)$，势函数 $V_{ij}(\xi)$ 对 t 是 T-周期且在 $\xi=0$ 奇异但满足 Gordon 的强力条件。证明基于 K. Uhlenbeck，丁伟岳及 P. Majer-S. Terracini 的扰动方法的一个变形。

关键词　N-体型问题；无穷多非碰撞周期解；Ljusternik-Schnirelmann 理论

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无穷多周期解

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周期解