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Topology of Isoenergy manifold of Natural Hamiltonian Systems and Existence of Global Poincaré Section*

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Abstract: The fundamental group of compact isoenergy manifolds of natural n-degree of freedom Hamiltonian systems is discussed. It is demonstrated that a certain topology of corresponding accessible region in configuration space gives rise to topological obstruction to the existence of global Poincaré sections

Key words: n-degree freedom; Hamiltonian Systems; Poincaré sections

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In the last decades, much progress has been made in topology of Hamiltonian systems. A Russian school headed by A. t. Fomenko[1], along the line initiated by S. Smale in his paper topology and mechanics[2], has systematically introduced the topological invariants for isoenergy mainfolds in integrable Hamiltonian systems with two degrees of freedom and obtained a topological classification of isoenergy manifolds in case of two degrees of freedom. Recently, A. Bolsinov, et al.[3], made an extensive investigation on topology of isoenergy manifolds of natural Hamiltonian systems with two degrees of freedom, giving a complete classification of 3-isoenergy manifolds in terms of Euler characteristic of the accessible region in configuration space and discussed the relation between topology of isoenergy mainifolds and the existence of global Poincaré sections(called complete section therein).

We investigate the topology of isoenergy of Hamiltonian systems with n-degree ($n>2$) of freedom and its restriction against existence of global Poincaré sections. For the sake of convenience, we only consider the situation where the phase is Euclidean space R^{2n}.

1 Fundamental group of isoenergy manifold of natural Hamiltonian systems

First, we consider on R^{2n} the Hamiltonian function of the form

$$H(x,y) = \frac{1}{2}\sum_{i}^{n} x_i^2 + V(y_1,\cdots,y_n)$$

where $(x,y)\in R^n \times R^n = R^{2n}$. The function with the above form are usually called natural Hamiltonian function([3]). Throughout this paper we assume that $H(x,y)$ is differentiable which clearly means that V is also differentiable.

Assume that c is a regular value of $H(x,y)$. It is evident that c is also the regular value of potential function $V(y)$. Assume also that $M = V^{-1}(-\infty, c]$ is a compact submanifold with nonempty boundary $\partial M = V^{-1}(c)$. In this section we investigate the relation of fundamental group of isoenergy manifold $H^{-1}(c)$ with that of M and ∂M.

Now let N be a sufficiently small collar neighborhood of ∂M in M, such that $N = V^{-1}(c-\delta, c)\cong\partial M\times[0,1]$ for a small positive number δ, where "$\cong$" means homeomorphism, $H(x,y)$ can be locally ex-

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pressed on $D^n \setminus N$ as

$$H(x,y) = \overline{H}(x,t) = \frac{1}{2}\sum_{i=1}^{n} x_i^2 + t, x \in D^n, t \in [0,1]$$

where $D^n \subset R^n$ is a small ball wth origin being the center. In view of this fact we can construct a covering $\{U_1, U_2\}$ of $H^{-1}(c)$ as follows:

For $\partial M \subset H^{-1}(c)$, let us consider a tubular neighborhood of ∂M in $H^{-1}(c)$. For the sake of convenience, denote by $D^n_\varepsilon \times \partial M$ this neighborhood, D^n_ε is an n-ball with radius ε. Now take another smaller tubular neighborhood of ∂M, $D^n_{\varepsilon/2} \times \partial M$. Let

$$U_1 = D^n_\varepsilon \times \partial M$$

$$U_2 = H^{-1}(c) - D^n_{\varepsilon/2} \times \partial M$$

Then $H^{-1}(c) \subseteq U_1 \cup U_2$, and in addition, we have

$$U_1 \cap U_2 = (D^n_\varepsilon - D^n_{\varepsilon/2}) \times \partial M \cong S^{n-1} \times [0,1] \times \partial M$$

Furthermore, it is easy to see that U_2 admits a natural trivial fiber bundle structure

$$U_2 = S^{n-1} \times M'$$

where $M' = M - [0, \varepsilon/2] \times \partial M$, which is evidently diffeomorphic to M.

For the fundamental group of $H^{-1}(c)$, we have the following theorem.

Theorem 1 The fundamental group $\pi_1(H^{-1}(c))$ is isomorphic to

$$\{\pi_1(\partial M) * [\pi_1(S^{n-1}) \times \pi_1(M)];$$
$$j_*(z) = k_*(z), \forall z \in [\pi_1(S^{n-1}) \times \pi_1(\partial M)]\}$$

where

$$j: U_1 \cap U_2 \to U_1$$

and

$$k: U_1 \cap U_2' \to U_2$$

are inclusion maps, respectively. The group $\pi_1(\partial M) * [\pi_1(S^{n-1}) \times \pi_1(M)]$ is the free product of $\pi_1(\partial M)$ and $\pi_1(S^{n-1}) \times \pi_1(M)$.

This theorem shows how the fundamental group of isoenergy manifold $H^{-1}(c)$ depends on the potential function $V(y)$. Before proving this theorem, we give some consequence of it.

Corollary 1 For a Hamiltonian systems H of more than two degrees of freedom, if c is a regular value of H, then

$$\pi_1(H^{-1}(c)) \approx \{\pi_1(\partial M) * \pi_1(M);$$
$$j_*(z) = k_*(z), \forall z \in \pi_1(\partial M)\}$$

Corollary 2 In case of M being simply connected and $n > 2$, $H^{-1}(c)$ is simply connected.

Proof It is clear in this case that $U_2 = S^{n-1} \times M'$ is simply connected. Therefore

$$k_*(z) = e, \forall 2 = \pi_1(\partial M)$$

Now it follows that from Theorem 1 that

$$\pi_1(H^{-1}(c) \approx \{\pi_1(\partial M);$$
$$j_*(z) = e, \forall z \in \pi_1(\partial M)\} \approx e$$

Evidently the following holds.

Corollary 3 If ∂M is simply connected and $n > 2$, then

$$\pi_1(H^{-1}(c)) \approx \{\pi_1(M)$$

Now we prove the main theorem.

Proof of Theorem 1 Applying Van Kampen theorem [4] to the covering pair $\{U_1, U_2\}$ yields

$$\pi_1(H^{-1}(c)) = \pi_1(U_1 \cup U_2) \approx$$
$$\pi_1(U_1) * \pi_1(U_2);$$
$$j_*(z) = k_*(z), \forall z \in [\pi_1(U_1 \cap U_2)\}$$

Keeping in mind that $U_1 \cong D^n_\varepsilon \times \partial M$, $U_2 \cong S^{n-1} \times M$ and $U_1 \cap U_2 \cong S^{n-1} \times [0,1] \times \partial M$, we have

$$\pi_1(U_1) = \pi_1(\partial M)$$

$$\pi_1(U_1 \cap U_2) = \pi_1(S^{n-1}) \times \pi_1(\partial M)$$

Therefore

$$\pi_1(H^{-1}(c)) \approx \{\pi_1(\partial M) * [\pi_1(S^{n-1}) \times \pi_1(M)]$$
$$j_*(z) = k_*(z), \forall z \in [\pi_1(S^{n-1}) \times \pi_1(M)]\}$$

2 Topological obstruction to global Poincaré section

In this section we will show how the fundamental group of isoenergy manifold of a Hamiltonian system affects the existence of global Poincaré section. First we recall the concept global Poincaré section[5].

Definition 3 Let M be a compact connected differentiable n dimensional manifold, and Ψ_t a smooth flow on it. Suppose that there exists a compact $n-1$ dimensional submanifold $N \subset M$ having the flowing properties:

(i) every trajectory has a point of transversal intersection with N;

(ii) for any point $x \in N$, there exists a minimal time $t_0 > 0$ such that $\Psi_{t_0} x \in N$.

Then N is called global Poincaré section for Ψ_t. For x

$\in N$, let $\Psi(x)$ be the first point of intersection of Ψ_t with N for $t>0$, we get a globally defined homeomorphism $\Psi: N \to N$, $\Psi(x)$ is called Poincaré section map.

The following known result [3,6] shows that the existence of global Poincaré section inposes a strict topological condition upon the phase space.

Lemma 1 If a flow on a manifold M has a global Poincare section, then M can admit a fiber bundle structure with S_1 being the base space. Furthermore, the fundamental group satisfies

$$\pi_1(M)/\pi_1(N) \approx Z$$

where N is the global Poincaré section and Z is the integer group.

From Lemma 1 and the discussion is Section 2, we obtain the following result.

Theorem 2 Consider on R^{2n} the natural Hamiltonian system of the form

$$H(x,y) = \frac{1}{2}\sum_{i=1}^{n} x_i^2 + V(y_1,\cdots,y_n)$$

where $(x,y)\in R^n \times R^n = R^{2n}$, $n>2$. Suppose that c is a regular value of $H(x,y)$, and the accessible configuration region $V^{-1}(-\infty, c]$ defined by the potential function $V(y)$ is simply connected. Then the Hamiltonian system H admits no global Poincaré section in the isoenergy manifold $H^{-1}(c)$.

Proof By the corollary 1, the condition that $V^{-1}(-\infty, c]$ is connected implies that $H^{-1}(c)$ is also simply connected. Therefore the fundamental group $\pi_1(H^{-1}(c))$ is trivial, this contradicts the assertion in lemma 1.

Remark In view of theorem 2, it can be seen that although Poincaré section is powerful tool for studying dynamical behavior in Hamiltonian system, it is not always possible to find a global Poincaré section.

References:

[1] FOMENKO A T. Topological classification of all integrable Hamiltonian differential equations of general type with two degrees of freedom [A]. Symposium on the Geometry of Hamiltonian system ed T. Ratiu [C]. New York: Springer, 1999.131-339.

[2] SMALE S. Topology and Mechanics [J]. I Invent Math, 1970, 10: 305-331

[3] BOLSINOV A. Topology of energy surfaces and existence of transversal Poincaré section [J]. J Phys A Math Gen. 1996, 29: 4 977-4 985

[4] STILLWELL J. Classical topology and combinatorial group theory [M]. New York: Springer-Verlag, 1980

[5] KUSNIRENKO A G. Problems in the general theory of dynamical systems on a manifold [J]. Amer Math Transl, 116 (2): 198.

[6] YANG X S. Toppological obstruction to existence of global Poincaré section [J]. Mathematic Applicata, 1999, 12(2): 72-75

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自然哈密尔顿系统能量面的拓扑与整体 Poincaré 截面的存在性

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摘　要: 讨论了 n-自由度自然哈密尔顿系统能量面的基本群。证明了构形空间中与能量面对应的可达区域的某种拓扑可以构成整体 Poincaré 截面存在性的拓扑障碍。

关键词: n-自由度; 哈密尔顿系统; Poincaré 截面 能量面, 存在性

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