文章编号:1000-582X(2005)07-0133-04

# 动态球对称黑洞中 Dirac 粒子的 Hawking 辐射*

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**摘 要:** 在动态球对称黑洞时空中求解狄拉克方程,采用了 Tortoise 坐标变换将狄拉克方程变成 Tortoise 坐标下的形式,在视界面附近化成了标准的波动方程,得到在视界面附近狄拉克粒子的 Hawking 辐射温度,成功地导出了 Hawking 热谱公式.该谱由黑洞的度规分量 $g_{00}$ 和 $g_{01}$ 决定.

**关键词:** 狄拉克方程;Hawking 辐射;黑洞;Tortoise 坐标变换

**中图分类号:** P145.8 **文献标识码:** A

自从霍金(Hawking)指出黑洞会发射热辐射以来,人们已经反复讨论了史瓦西黑洞、克尔-纽曼黑洞、电磁直线加速动态黑洞等黑洞的热辐射.研究的方法有用辐射反作用研究动态黑洞的热辐射、用共形平直技术研究动态黑洞的热效应、用旋量方程研究动态黑洞狄拉克粒子的热辐射等等.文献[1]研究了动态球对称黑洞的 Klein—Gordon 粒子霍金热辐射.文中用旋量方程研究动态球对称黑洞中狄拉克粒子的热辐射,首先构造一套零标架、用这套零标架计算旋系数、将弯曲时空的狄拉克方程退耦为普通导数表示的偏微分方程、作 Tortoise 坐标变换、在视界面附近解狄拉克方程、得到所期望的霍金热辐射谱公式.

## 1 动态球对称黑洞的线元及零标架的选取

动态球对称时空的线元为[2]:

$$ds^2 = e^{2\psi}(1-\frac{2m}{r})dv^2 - 2e^{\psi}dvdr - r^2(d\theta^2+\sin^2\theta d\varphi), \quad (1)$$

其中 $m=m(v,r)$, $\psi=\psi(v,r)$,采用的号差为[+ - - -],由度规(1)构造如下零标架的协变分量为

$$l_\mu = [-e^{\psi}\ 0\ 0\ 0],\ n_\mu = \left[-\frac{1}{2}e^{\psi}(1-\frac{2m}{r})\ 1\ 0\ 0\right],$$

$$m_\mu = \frac{r}{\sqrt{2}}[0\ 0\ 1\ -i\sin\theta],\ \overline{m}_\mu = \frac{r}{\sqrt{2}}[0\ 0\ 1\ i\sin\theta]. \quad (2)$$

由式(2)和度规可得零标架的逆变分量:

$$l^\mu = [0\ 1\ 0\ 0],\ n^\mu = \left[-e^{-\psi}\ -\frac{1}{2}(1-\frac{2m}{r})\ 0\ 0\right],$$

$$m^\mu = \frac{1}{\sqrt{2}r}\left[0\ 0\ -1\ \frac{i}{\sin\theta}\right],\ \overline{m}^\mu = \frac{1}{\sqrt{2}r}\left[0\ 0\ -1\ \frac{-i}{\sin\theta}\right]. \quad (3)$$

不难证明,式(2)、(3)满足零标架的定义式(见文献[3])

$$l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = \overline{m}_\mu m^\mu = 0,$$
$$l_\mu n^\mu = -m_\mu \overline{m}^\mu = 1,$$
$$l_\mu m^\mu = l_\mu \overline{m}^\mu = n_\mu m^\mu = n_\mu \overline{m}^\mu = 0,$$
$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \overline{m}_\nu - \overline{m}_\mu m_\nu. \quad (4)$$

由式(2)、(3),按照 Newman 和 Penrose[4] 文献的(4.1a)式计算旋系数如下:

$$\rho = -\frac{1}{r},\ \mu = -\frac{1}{2r^2}(r-2m),\ \alpha = \frac{1}{2\sqrt{2}r}\mathrm{ctg}\theta,$$

$$\varepsilon = \frac{1}{2}\dot{\psi},\ \gamma = \frac{1}{4}[(1-\frac{2m}{r})\psi' + \frac{2m}{r^2} - \frac{2}{r}m'],$$

$$\beta = -\alpha,\ \kappa = \sigma = \tau = \lambda = \pi = \upsilon = 0. \quad (5)$$

式中 $\dot{\psi} = \frac{\partial\psi}{\partial v}$, $\psi' = \frac{\partial\psi}{\partial r}$, $m' = \frac{\partial m}{\partial r}$.

## 2 狄拉克方程

弯曲时空的狄拉克方程为[4]:

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* 收稿日期:2005-02-28
基金项目:国家自然科学基金资助项目(10373003);重庆市教委资助项目(031103)
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$$(D+\varepsilon-\rho)F_1+(\bar{\delta}+\pi-\alpha)F_2-\frac{1}{\sqrt{2}}i\mu_0 G_1=0,$$
$$(\Delta+\mu-\gamma)F_2+(\delta+\beta-\tau)F_1-\frac{1}{\sqrt{2}}i\mu_0 G_2=0,$$
$$(D+\varepsilon^*-\rho^*)G_2-(\delta+\pi^*-\alpha^*)G_1-\frac{1}{\sqrt{2}}i\mu_0 F_2=0,$$
$$(\Delta+\mu^*-\rho^*)G_1-(\bar{\delta}+\beta^*-\tau^*)G_2-\frac{1}{\sqrt{2}}i\mu_0 F_1=0. \tag{6}$$

式中：$D=\partial_{0\dot{0}}=l^\mu\partial_\mu,\quad \Delta=\partial_{1\dot{1}}=n^\mu\partial_\mu,$

$$\delta=\partial_{0\dot{1}}=m^\mu\partial_\mu,\ \bar{\delta}=\partial_{1\dot{0}}=\bar{m}^\mu\partial_\mu. \tag{7}$$

$\mu_0$ 是 Dirac 粒子的静止质量. 由式(3)、(7)可得：

$$D=\frac{\partial}{\partial r},\ \Delta=-e^{-\psi}\frac{\partial}{\partial v}-\frac{1}{2}(1-\frac{2m}{r})\frac{\partial}{\partial r},$$
$$\delta=-\frac{1}{\sqrt{2}r}[\frac{\partial}{\partial\theta}-\frac{i}{\sin}\frac{\partial}{\partial\varphi}],\ \bar{\delta}=-\frac{1}{\sqrt{2}r}[\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}]. \tag{8}$$

将式(5)、(8)代入式(6)，可得：

$$\sqrt{2}rD^+F_1-L^+F_2=i\mu_0 rG_1,\ \sqrt{2}rD_-F_2+\bar{L}F_1=-i\mu_0 rG_2,$$
$$\sqrt{2}rD^+G_2+\bar{L}G_1=i\mu_0 rF_2,\ \sqrt{2}rD_-G_1-L^+G_2=-i\mu_0 rF_1. \tag{9}$$

式中

$$D^+=\frac{\partial}{\partial r}+\frac{1}{2}\psi'+\frac{1}{r},$$
$$D_-=e^{-\psi}\frac{\partial}{\partial v}+\frac{1}{2}(1-\frac{2m}{r})\frac{\partial}{\partial r}+\frac{1}{2r}(1-\frac{m}{r})+\frac{1}{4}(1-\frac{2m}{r})\psi'-\frac{1}{2r}m',$$
$$L^+=\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}+\frac{1}{2}\mathrm{ctg}\theta,$$
$$\bar{L}=\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}+\frac{1}{2}\mathrm{ctg}\theta. \tag{10}$$

分离变量，设：

$$F_1(v,r,\theta)=R_-(v,r)S_-(\theta,\varphi),$$
$$F_2(v,r,\theta)=R_+(v,r)S_+(\theta,\varphi),$$
$$G_1(v,r,\theta)=R_+(v,r)S_-(\theta,\varphi),$$
$$G_2(v,r,\theta)=R_-(v,r)S_+(\theta,\varphi). \tag{11}$$

则方程(9)变为：

$$\sqrt{2}rD^+R_-=(\lambda+i\mu_0 r)R_+,\ \sqrt{2}rD_-R_+=(\lambda-i\mu_0 r)R_-,$$
$$L^+S_+=\lambda S_-,\ \bar{L}S_-=-\lambda S_+. \tag{12}$$

式中，$\lambda$ 是分离变量引入的常数. 进一步从式(12)中可以得到关于 $R_+(v,r)$、$R_-(v,r)$、$S_+(\theta,\varphi)$、$S_-(\theta,\varphi)$ 的退耦 Dirac 方程，而我们所关心的是狄拉克方程的径向部分，由式(12)不难得到：

$$2rD^+\left(\frac{r}{\lambda-i\mu_0 r}D_-R_+\right)=(\lambda+i\mu_0 r)R_+,$$
$$2rD_-\left(\frac{r}{\lambda+i\mu_0 r}D^+R_-\right)=(\lambda-i\mu_0 r)R_-. \tag{13}$$

考虑到式(10)得：

$$e^{\psi}(1-\frac{2m}{r})\frac{\partial^2 R_+}{\partial r^2}+2\frac{\partial^2 R_+}{\partial r\partial v}+(\frac{4}{r}-\psi'+\frac{i2\mu_0}{\lambda-i\mu_0 r})\frac{\partial R_+}{\partial v}+e^{\psi}[(1-\frac{2m}{r})\psi'-\frac{3}{r}m'+\frac{3}{r^2}(r-m)+\frac{i\mu_0}{\lambda-i\mu_0 r}(1-\frac{2m}{r})]\frac{\partial R_+}{\partial r}+B_+R_+=0. \tag{14}$$

式中：

$$B_+=e^{\psi}\{(1-\frac{2m}{r})[\frac{1}{4}\psi'^2+\frac{1}{2}\psi'']-\frac{2}{r^2}m'-\frac{1}{r}m''+\frac{1}{r^2}+\frac{3}{2r^2}(r-m)\psi'+\frac{1}{r^2}(\lambda^2+\mu_0^2r^2)\}+\frac{i\mu_0}{\lambda-i\mu_0 r}[\frac{1}{r}(1-\frac{m}{r})+\frac{1}{2}(1-\frac{2m}{r})\psi'-\frac{1}{r}m']e^{\psi}. \tag{15}$$

其中 $\psi''=\frac{\partial^2\psi}{\partial r^2}$，$m''=\frac{\partial^2 m}{\partial r^2}$. 类似可得 $R_-$ 的方程.

## 3 作 Tortoise 坐标变换

作 Tortoise 坐标变换，令：

$$r_*=r+\frac{1}{2\kappa(v_0)}\ln(r-r_H(v)),\ v_*=v \tag{16}$$

式中 $r_H$ 是视界的位置有：

$$\frac{\partial}{\partial r}=\frac{2\kappa(r-r_H)+1}{2\kappa(r-r_H)}\frac{\partial}{\partial r_*},\ \frac{\partial}{\partial v}=\frac{\partial}{\partial v_*}-\frac{\dot{r}_H}{2\kappa(r-r_H)}\frac{\partial}{\partial r_*},$$
$$\frac{\partial^2}{\partial r^2}=[\frac{2\kappa(r-r_H)+1}{2\kappa(r-r_H)}]^2\frac{\partial^2}{\partial r_*^2}-\frac{1}{2\kappa(r-r_H)^2}\frac{\partial}{\partial r_*},$$
$$\frac{\partial^2}{\partial r\partial v}=-\frac{[2\kappa(r-r_H)+1]\dot{r}_H}{[2\kappa(r-r_H)]^2}\frac{\partial^2}{\partial r_*^2}+\frac{\dot{r}_H}{2\kappa(r-r_H)}\frac{\partial}{\partial r_*}+\frac{2\kappa(r-r_H)+1}{2\kappa(r-r_H)}\frac{\partial^2}{\partial r_*\partial v_*}. \tag{17}$$

代入式(14)并以$\frac{2\kappa(r-r_H)}{2\kappa(r-r_H)+1}$相乘整理后得

$$\frac{[2\kappa(r-r_H)+1](r-2m)e^{\psi}-2r\dot{r}_H}{2\kappa r(r-r_H)}\frac{\partial^2 R_+}{\partial r_*^2}+2\frac{\partial^2 R_+}{\partial r_*\partial v_*}+[\frac{2r\dot{r}_H-(r-2m)e^{\psi}}{r(r-r_H)[2\kappa(r-r_H)+1]}+(1-\frac{2m}{r})\psi' e^{\psi}-\frac{3}{r}m' e^{\psi}+\frac{3}{r^2}(r-m)e^{\psi}+$$

$$\frac{i\mu_0}{\lambda - i\mu_0 r}(1 - \frac{2m}{r})e^{\psi} -$$
$$\frac{\dot{r}_H}{2\kappa(r - r_H) + 1}(\frac{4}{r} - \psi' + \frac{i2\mu_0}{\lambda - i\mu_0 r})]\frac{\partial R_+}{\partial r_*} +$$
$$\frac{2\kappa(r - r_H)}{2\kappa(r - r_H) + 1}(\frac{4}{r} - \psi' + \frac{i2\mu_0}{\lambda - i\mu_0 r})\frac{\partial R_+}{\partial v_*} +$$
$$\frac{2\kappa(r - r_H)}{2\kappa(r - r_H) + 1}B_+ R_+ = 0 . \quad (18)$$

## 4 视界位置及辐射谱

在式(1)时空中,表征事件视的零曲面方程为:

$$g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} = 0 . \quad (19)$$

考虑到时空具有球对称,视界作为特征曲面应该有:$f=f(v,r)$,将式(1)代入上式得:

$$-2e^{-\psi}\frac{\partial f}{\partial v}\frac{\partial f}{\partial r} - (1 - \frac{m}{r})(\frac{\partial f}{\partial r})^2 = 0 . \quad (20)$$

作为零超曲面有$f=f(v,r)=0$,这样就确定了事件视界面的位置:

$$r_H = \frac{2M}{1 - 2\dot{r}_H e^{-\psi}} . \quad (21)$$

其中$M=M(r_H,v)$. 从式(21)容易看出,式(18)第一项系数的分子在$r \to r_H$时为零,即

$$\lim_{r \to r_H}\{[2\kappa(r - r_H) + 1](r - 2m)e^{\psi} - 2r\dot{r}_H\} = 0 . \quad (22)$$

所以在$r \to r_H$的极限情况下,该系数的极限值为

$$\eta = \lim_{r \to r_H}\frac{[2\kappa(r - r_H) + 1](r - 2m)e^{\psi} - 2r\dot{r}_H}{2\kappa r(r - r_H)} = \frac{1 + 2\kappa(r_H - 2M) - 2m' + (r_H - 2M)\psi' - 2\dot{r}_H}{2\kappa r_H e^{-\psi}} . \quad (23)$$

希望式(18)在视界附近趋于标准的波动方程,这就要求$\eta = 1$. 选择调节参数$\kappa$令其等于:

$$\kappa = \frac{1 - 2m' - 2\dot{r}_H e^{-\psi}(1 - r_H\psi')}{4M + 2r_H(e^{-\psi} - 1)} . \quad (24)$$

此时$\frac{\partial R_+}{\partial r_*}$的系数在$r \to r_H$时的极限值为

$$A = \frac{1}{r_H}(1 + \frac{M}{r_H})e^{\psi} + 3\dot{r}_H\psi' - \frac{3}{r_H}m'e^{\psi} . \quad (25)$$

而$\frac{\partial R_+}{\partial v_*}$和$R_+$的系数在$r \to r_H$时趋于0,所以在视界面附近式(18)化成:

$$\frac{\partial^2 R_+}{\partial r_*^2} + 2\frac{\partial^2 R_+}{\partial r_* \partial v_*} + A\frac{\partial R_+}{\partial r_*} = 0 . \quad (26)$$

分离变量,得到满足式(26)的径向入射波解和出射波解:

$$R_+^{in} = e^{-i\omega v_*} , R_+^{out} = e^{-i\omega v_*}e^{i2\omega r_*}e^{-Ar_*} . \quad (27)$$

在视界附近有:

$$r_H \sim \frac{1}{2\kappa}\ln(r - r_H) . \quad (28)$$

出射波在视界外又可写成:

$$R_+^{out} = e^{-i\omega v_*}e^{i2\omega r}(r - r_H)^{i\omega/\kappa}(r - r_H)^{-A/2\kappa} . \quad (29)$$

此波函数在视界$r = r_H$处非解析,只能通过下半复平面把其解析延拓到视界内,即:

$$r - r_H \to |r - r_H| e^{-i\pi} = (r_H - r)e^{-i\pi} . \quad (30)$$

于是,在视界内出射波为

$$\tilde{R}_+^{out} = e^{-i\omega v_*}e^{i2\omega r}(r - r_H)^{i\omega/\kappa}e^{\pi\omega/\kappa}e^{i\pi A/2\kappa}(r - r_H)^{-A/2\kappa} = e^{-i\omega v_*}e^{i2\omega r_*}e^{-Ar_*}e^{\pi\omega/\kappa}e^{i\pi A/2\kappa} . \quad (31)$$

出射波在视界面上的散射几率为:

$$\left|\frac{R_+^{out}}{\tilde{R}_+^{out}}\right|^2 = e^{-2\pi\omega/\kappa} . \quad (32)$$

用 Damour, Ruffini[5]和 Sannan[6]方法不难得到

$$N_\omega^2 = \frac{1}{e^{\omega/k_B T} + 1} . \quad (33)$$

式中,$T = \frac{\kappa}{2\pi k_B}$为黑洞辐射温度;$k_B$为玻耳兹曼常数,$\kappa$由式(24)给出,$\kappa$是辐射温度函数.

## 5 结果讨论

式(33)就是动态球对称黑洞在视界面上狄拉克粒子的 Hawking 辐射的热谱公式,在此公式中包含的参数$T$由黑洞的度规分量$e^{2\psi}(1 - \frac{2m}{r})$和$-e^{\psi}$决定.

同理,对$R_-$可以得到相同的结论.

如果由度规(1)构造另外一套零标架的协变分量

$$l_\mu = \left[\frac{1}{2}e^{2\psi}(1 - \frac{2m}{r}) \ - e^{\psi} \ 0 \ 0\right], n_\mu = [1\ 0\ 0\ 0],$$
$$m_\mu = \frac{r}{\sqrt{2}}[0\ 0\ 1\ i\sin\theta], \bar{m}_\mu = \frac{r}{\sqrt{2}}[0\ 0\ 1\ -i\sin\theta] . \quad (34)$$

采用与前面类似的计算过程,同相可在视界附近将狄拉克方程化为标准的波动方程,得到与式(24)完全相同的辐射温度函数$\kappa$. 这说明在同一度规下可选取不同的零标架,但狄拉克粒子的 Hawking 辐射谱是相同的.

总之,求解动态球对称黑洞中狄拉克方程,在视界处得到了标准形式的出射波函数,得到决定事件视界

位置和狄拉克粒子的 Hawking 辐射谱. 此结果与研究标量粒子的一致. 可见,在动态球对称黑洞在视界面附近辐射的能谱具有黑体辐射的形式.

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# Hawking Radiation of Charged Dira Partices in Non-static and Spherically Symmetric Black Hole


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**Abstract**: The thesis studies the solution of Dirac equation in the non-static and spherically symmetric black hole, it change into the form of tortoise coordinate by employing the tortoise coordinate transformation. Then this changes into standard wavw equation near the event horizons and gwts Hawking radiation temperature of Dirac particles near the event horizons, which successfully obtains Hawking thermal spectrum formula. Such thermal spectrum is determined by black holes metric tensor hefts $g_{00}$ and $g_{01}$.



**Key words**: dirac equation; hawking radiation; black hole; tortoise coordinate transformation



(编辑 陈移峰)