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双极化态柱面引力波的能量动量赝张量*

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摘 要:利用 Rosen-virbhadra(R-V)给出的引力场能量动量赝张量的一般表达式,计算得到了双极化态柱面引力波能量动量的具体形式,并讨论了其正定性问题与渐进行为. 发现:R-V 的笛卡尔坐标形式是对双极化态的柱面引力波能量动量更为合理的表述.

关键词:Rosen-virbhadra;能量动量赝张量;双极化态;柱面引力波
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1 双极化态柱面引力波在笛卡尔坐标系的度规

无论是在 Einstein 的广义相对论中,还是在其他相对论引力理论中,对引力场的能量动量的表述和计算都是非常重要的,同时也是非常困难的. 目前,单极化态的柱面引力波的能量动量赝张量已经在不同的坐标系中被计算过,它的合理性也已经被详细的讨论过了[1-3]. 从这些讨论中我们得知,与柱坐标系相比较,笛卡尔坐标系是一个描述单极化态柱面引力波的更为合理的坐标系,但是目前关于双极化态的柱面引力波的研究还比较少.

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - d\rho^2) - e^{2\psi}(dz + \omega d\phi)^2 - \rho^2 e^{-2\psi} d\phi^2 \qquad (1)$$

是我们求解双极化态柱面引力波的能量动量赝张量的出发点[4]. 其中 $\gamma = (\rho,t)$, $\psi = \psi(\rho,t)$, $\omega = \omega(\rho,t)$, 该线元描述的度规称为双极化态度规,所谓的双极化是指 ψ 极化和 ω 极化,亦即"+"型极化和"×"型极化. 显然该度规空间具有对称性. 由式(1)我们可以很容易地写出柱坐标系中度规的协变形式如下:

$$g_{00} = e^{2(\gamma-\psi)},\quad g_{11} = e^{2(\gamma-\psi)},$$
$$g_{22} = -(e^{2\psi}\omega^2 + e^{-2\psi}\rho^2),$$
$$g_{23} = -e^{2\psi}\omega,\quad g_{33} = -e^{2\psi}. \qquad (2)$$

其逆变形式为:

$$g^{00} = e^{-2(\gamma-\psi)},\quad g^{11} = e^{-2(\gamma-\psi)},$$
$$g^{22} = -\rho^{-2}e^{2\psi},\quad g^{23} = -e^{2\psi}\omega\rho^{-2},$$
$$g^{33} = -e^{-2\psi} - \omega^2\rho^{-2}e^{2\psi}. \qquad (3)$$

则可得出行列式值为 $g = -e^{4(\gamma-\psi)}\rho^2$.

在式(1)中,用笛卡尔坐标系中的参数 x, y 取代柱坐标系中 ρ, ϕ, 利用 $x = \rho\cos\phi, y = \rho\sin\phi, z = z$, 得出:

$$ds^2 = e^{2(\gamma-\psi)}dt^2 - [e^{2(\gamma-\psi)}\frac{x^2}{\rho^2} + e^{2\psi}\frac{\omega^2y^2}{\rho^4} + e^{-2\psi}\frac{y^2}{\rho^2}]dx^2 - 2[e^{2(\gamma-\psi)}\frac{xy}{\rho^2} - e^{2\psi}\frac{\omega^2xy}{\rho^4} - e^{-2\psi}\frac{xy}{\rho^2}]dxdy - [e^{2(\gamma-\psi)}\frac{y^2}{\rho^2} + e^{2\psi}\frac{\omega^2x^2}{\rho^4} + e^{-2\psi}\frac{x^2}{\rho^2} -]dy^2 - 2e^{2\psi}(\frac{\omega x}{\rho^2})dydz + 2e^{2\psi}(\frac{\omega y}{\rho^2})dxdz - e^{2\psi}dz^2, \qquad (4)$$

由式(4)可以写出笛卡尔坐标系中度规的协变形式如下:

$$g_{00} = e^{2(\gamma-\psi)}, g_{11} = -e^{-2\psi}(e^{2\gamma}x^2 + e^{4\psi}y^2\omega^2/\rho^2 + y^2)/\rho^2,$$
$$g_{12} = -e^{-2\psi}(e^{2\gamma} - e^{4\psi}\omega^2/\rho^2 - 1)xy/\rho^2,$$
$$g_{22} = -e^{-2\psi}(e^{2\gamma}y^2 + e^{4\psi}x^2\omega^2/\rho^2 + x^2)/\rho^2,$$
$$g_{13} = e^{2\psi}y\omega/\rho^2, g_{23} = -e^{2\psi}x\omega/\rho^2, g_{33} = -e^{2\psi}.$$

下面利用坐标变换法求笛卡尔坐标系下的逆变度规,根据 $g^{ij} = \frac{\partial x^i \partial x^j}{\partial \overline{x^m} \partial \overline{x^n}}\overline{g^{mn}}$ (其中 $x^0 = t, x^1 = x, x^2 = y, x^3 = z, \overline{x^0} = t, \overline{x^1} = \rho, \overline{x^2} = \phi, \overline{x^3} = z$, $\overline{g^{mn}}$ 为柱坐标系

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下的逆变度规张量）．我们可以求得笛卡尔坐标系下的逆变度规张量为：

$$
\begin{aligned}
g^{00} &= e^{2(\psi-\gamma)}, g^{11} = -e^{2\psi}(e^{-2\gamma}x^2 + y^2)/\rho^2, \\
g^{12} &= e^{2\psi}(1 - e^{-2\gamma})xy/\rho^2, \\
g^{22} &= -e^{2\psi}(e^{-2\gamma}y^2 + x^2)/\rho^2, \\
g^{13} &= -e^{2\psi}y\omega/\rho, g^{23} = e^{2\psi}x\omega/\rho^2, \\
g^{33} &= -e^{-2\psi} - e^{2\psi}\omega^2/\rho^2.
\end{aligned} \tag{6}
$$

可以计算出度规行列式为 $g = -e^{4(\gamma-\varphi)}$.

2　双极化态柱面引力波在笛卡尔坐标系下能量动量流密度

一般的，Einstein 场方程可写为 $G_{\mu\nu} = -8\pi T_{\mu\nu}$ ，其中 $T_{\mu\nu}$ 是物质的能量动量赝张量，满足 $T^{\nu}_{\mu;\nu} = 0$ ．能量动量张量的表示采用 Rosen - Virbhadra 的表述形式：$\sqrt{-g}\Lambda^{\nu}_{\mu} = \sqrt{-g}T^{\nu}_{\mu} + \theta^{\nu}_{\mu} = \frac{1}{16\pi}H^{\nu\sigma}_{\mu,\sigma}$（右下角标表示对逗号后的指标求一次偏导）．在真空中 $T^{\nu}_{\mu} = 0$ ，则有

$$
\theta^{\nu}_{\mu} = \frac{1}{16\pi}H^{\nu\sigma}_{\mu,\sigma} = \frac{1}{16\pi}\{(-g)^{-1/2}g_{\mu\lambda}[-g(g^{\nu\lambda}g^{\sigma\tau} - g^{\sigma\lambda}g^{\nu\tau})]_{,\tau}\}_{,\sigma}, \tag{7}
$$

将柱坐标系下的度规表达式(2)、(3)代入式(7)，有 $\theta^{\nu}_{0} = \theta^{0}_{\nu} = 0$，其中 $\mu,\nu = 0,1,2,3$ ，即柱坐标系下的双极化态柱面引力波的能量动量密度为 0. 动量流密度分量为（中间的计算过程不在此罗列）：

$$
\begin{aligned}
&\theta^1_1 = \theta^2_1 = \theta^1_2 = \theta^3_1 = \theta^1_3 = 0, \\
&\theta^2_2 = \frac{1}{16\pi}[(2\gamma_t\rho - e^{4\psi}\omega\omega_t/\rho)_t + (e^{4\psi}\omega\omega_\rho/\rho - 2\gamma_\rho\rho)_\rho], \\
&\theta^2_3 = \frac{1}{16\pi}[(-e^{4\psi}\omega_t/\rho)_t + (e^{4\psi}\omega_\rho/\rho)_\rho], \\
&\theta^3_2 = \frac{1}{16\pi}[(e^{4\psi}\omega^2\omega_t/\rho - \rho\omega_t - 4\omega_\rho\psi_t)_t + (-e^{4\psi}\omega^2\omega_\rho/\rho + \rho\omega_\rho + 4\omega_\rho\psi_\rho)_\rho], \\
&\theta^3_3 = \frac{1}{16\pi}[(e^{4\psi}\omega\omega_t/\rho + 2\gamma_t\rho - 4\psi_t\rho)_t - (e^{4\psi}\omega\omega_\rho/\rho - 2\gamma_\rho\rho + 4\psi_\rho\rho)_\rho.
\end{aligned} \tag{8}
$$

其中，等式右边的右下角标表示对该指标求一次偏导，由上面的结果可以看出，在柱坐标系中，双极化态柱面引力波的动量流密度不恒为 0. 下面我们来计算笛卡尔坐标系中双极化态柱面引力波的能量动量赝张量．把笛卡尔坐标系下的度规表达式(5)、(6)代入 $H^{\nu\sigma}_{\mu} = (-g)^{-1/2}g_{\mu\lambda}[-g(g^{\nu\lambda}g^{\sigma\tau} - g^{\sigma\lambda}g^{\nu\tau})]_{,\tau}$ [5]有

$$
\begin{aligned}
&H^{00}_0 = 0, H^{01}_0 = (e^{2\gamma} - 1)x/\rho^2, \\
&H^{02}_0 = (e^{2\gamma} - 1)y/\rho^2, \\
&H^{03}_0 = 0, H^{00}_1 = 0, \\
&H^{01}_1 = \omega\omega_t y^2 e^{4\psi} - 2\gamma_t y^2/\rho^2, \\
&H^{02}_1 = 2\gamma_t xy/\rho^2 - \omega\omega_t xye^{4\psi}, H^{00}_2 = 0, \\
&H^{01}_2 = 2\gamma_t xy/\rho^2 - \omega\omega_t xye^{4\psi}, \\
&H^{02}_2 = \omega\omega_t x^2 e^{4\psi} - 2\gamma_t x^2/\rho^2, \\
&H^{00}_3 = 0, H^{01}_3 = -\omega_t ye^{4\psi}/\rho^2, \\
&H^{02}_3 = -\omega_t xe^{4\psi}/\rho^2, \\
&H^{10}_0 = -(e^{2\gamma} - 1)x/\rho^2, \\
&H^{20}_0 = -(e^{2\gamma} - 1)y/\rho^2, H^{30}_0 = 0.
\end{aligned} \tag{9}
$$

将以上结果代入式(7)，经过进一步的计算可以得出：

$$
\begin{aligned}
&\theta^0_0 = \frac{1}{8\pi}e^{2\gamma}(\psi_t^2 + \psi_\rho^2) + \frac{1}{32\pi\rho^2}e^{4\psi+2\gamma}(\omega_t^2 + \omega_\rho^2), \\
&\theta^0_1 = \frac{x\psi_\rho\psi_t}{4\pi\rho} + \frac{e^{4\psi}}{16\pi\rho^3}x\omega_t\omega_\rho - \frac{\omega\omega_t x}{16\pi\rho^4}e^{4\psi}, \\
&\theta^0_2 = \frac{y\psi_\rho\psi_t}{4\pi\rho} + \frac{e^{4\psi}}{16\pi\rho^3}y\omega_t\omega_\rho - \frac{\omega\omega_t y}{16\pi\rho^4}e^{4\psi}, \\
&\theta^0_3 = \theta^3_0 = 0, \\
&\theta^1_0 = -e^{2\gamma}\left(\frac{x\psi_\rho\psi_t}{4\pi\rho} + \frac{e^{4\psi}}{16\pi\rho^3}x\omega_t\omega_\rho\right), \\
&\theta^2_0 = -e^{2\gamma}\left(\frac{y\psi_\rho\psi_t}{4\pi\rho} + \frac{e^{4\psi}}{16\pi\rho^3}y\omega_t\omega_\rho\right).
\end{aligned} \tag{10}
$$

根据真空中 Einstein 场方程 $R_{\mu\nu} = 0$ 可导出 γ,ψ,ω 之间的关系式为：

$$
\begin{aligned}
\gamma_\rho &= \rho(\psi_t^2 + \psi_\rho^2) + \frac{e^{4\psi}}{4\rho}(\omega_t^2 + \omega_\rho^2), \\
\gamma_t &= 2\rho\psi_\rho\psi_t + \frac{e^{4\psi}}{2\rho}\omega_t\omega_\rho.
\end{aligned} \tag{11}
$$

经过进一步的计算，可得动量流密度分量：

$$
\begin{aligned}
&\theta^1_2 = \frac{1}{16\pi}\Big[2\sin\phi\cos\phi\left(-\gamma_{tt} + \gamma_{\rho\rho} - \frac{\gamma_\rho}{\rho}\right) + e^{4\psi}\frac{\sin\phi\cos\phi}{\rho^2}\left(\frac{3\omega\omega_\rho}{\rho} - \omega_\rho^2 - 4\psi_\rho\omega\omega_\rho - \omega\omega_{\rho\rho}\right)\Big], \\
&\theta^2_1 = \theta^1_2, \\
&\theta^1_1 = \frac{1}{16\pi}\Big[2\gamma_{tt}\sin^2\phi - 2\gamma_{\rho\rho}\sin^2\phi - 2\gamma_\rho\frac{\cos^2\phi}{\rho} + \sin^2\phi\left(e^{4\psi}\frac{\omega}{\rho^2}\omega_\rho\right)_\rho + e^{4\psi}\frac{\omega}{\rho^3}\omega_\rho\cos^2\phi)\Big], \\
&\theta^2_2 = \frac{1}{16\pi}\Big[2\gamma_{tt}\cos^2\phi - 2\gamma_{\rho\rho}\cos^2\phi - 2\gamma_\rho\frac{\sin^2\phi}{\rho} +
\end{aligned}
$$

$$\cos^2\phi(e^{4\psi}\frac{\omega}{\rho^2}\omega_\rho)_\rho + e^{4\psi}\frac{\omega}{\rho^3}\omega_\rho\sin^2\phi)\Big],$$

$$\theta_3^3 = \frac{1}{16\pi}\Big[4e^{4\psi}\psi_t\omega\omega_t + e^{4\psi}\omega_t^2 + e^{4\psi}\omega\omega_{tt})/\rho^2 + 2\gamma_{tt} - 4\psi_{tt} + (\frac{e^{2\gamma}-1}{\rho} - 2\gamma_\rho + 4\psi_\rho - e^{4\psi}\frac{\omega}{\rho^2}\omega_\rho)_\rho + (\frac{e^{2\gamma}-1}{\rho} - 2\gamma_\rho + 4\psi_\rho - e^{4\psi}\frac{\omega}{\rho^2}\omega_\rho)/\rho,$$

$$\theta_3^1 = \frac{1}{16\pi}\Big[\frac{e^{4\psi}}{\rho}\sin\phi(\frac{\omega_\rho}{\rho} - 4\omega_\rho\psi_\rho - \omega_{\rho\rho})\Big],$$

$$\theta_3^2 = \frac{1}{16\pi}\Big[\frac{e^{4\psi}}{\rho}\cos\phi(\frac{\omega_\rho}{\rho} - 4\omega_\rho\psi_\rho - \omega_{\rho\rho})\Big],$$

$$\theta_1^3 = \frac{1}{16\pi}\Big\{\Big[\frac{\sin\phi}{\rho}(\omega_t + 4\omega\psi_t - e^{4\psi}\frac{\omega^2}{\rho^2}\omega_t)\Big]_t + \sin\phi\Big[(e^{4\psi}\frac{\omega^2}{\rho^3} - \frac{1}{\rho})\omega_\rho + 2\frac{\omega}{\rho^2} - 4\frac{\omega}{\rho}\psi_\rho - e^{2\gamma}\frac{\omega}{\rho^2}\Big]_\rho + \sin\phi\Big[\Big(e^{4\psi}\frac{\omega^2}{\rho^2} - 1\Big)\omega_\rho + 2\frac{\omega}{\rho} - 4\omega\psi_\rho\Big]\Big\},$$

$$\theta_2^3 = \frac{\cos\phi}{\sin\phi}\theta_1^3.$$

以上的计算结果表明,笛卡尔坐标系能更为合理地描述能量动量赝张量. 在柱坐标系中,双极化态柱面引力波的能量动量流密度为 0,这是因为能量动量赝张量是一个赝张量,而柱坐标系不是一个好的坐标系(即物理上有意义的坐标系);相反,笛卡尔坐标系却是一个描述柱面引力波度规的较为合理的坐标系.

3 能量动量流密合理性的讨论

下面我们将要讨论其合理性,由式(10)、式(11)可知

$$\theta_0^0 = \frac{1}{8\pi}e^{2\gamma}(\psi_t^2 + \psi_\rho^2) + \frac{1}{32\pi\rho^2}e^{4\psi+2\gamma}(\omega_t^2 + \omega_\rho^2) = \frac{\gamma_\rho}{8\pi\rho}e^{2\gamma} \quad (12)$$

容易得出 θ_0^0 的正定性始终满足. 双极化态的 Einstein 场方程的孤子解为:

$$e^{2\gamma} = \frac{b\rho^{\frac{1}{2}}}{(t^2-\rho^2)^{\frac{1}{2}}}, e^{2\psi} = \frac{w(a^2+1)}{(a^2+w^2)}, \quad \omega = \frac{-2a(t^2-\rho^2)^{\frac{1}{2}}}{(a^2+1)}. \quad (13)$$

(其中 a,b 为任意常数, $w = \frac{t-(t^2-\rho^2)^{\frac{1}{2}}}{\rho}$ 的解要求 $t \geq \rho$). 对式(13)左右两边求导,经计算可得:

$$\gamma_\rho = \frac{t^2+\rho^2}{4\rho(t^2-\rho^2)},$$

$$\gamma_t = \frac{-t}{2(t^2-\rho^2)},$$

$$\omega_\rho = \frac{-2a\rho}{(a^2+1)(t^2-\rho^2)^{\frac{1}{2}}},$$

$$\omega_t = \frac{-2at}{(a^2+1)(t^2-\rho^2)^{\frac{1}{2}}}.$$

θ_0^0 的正定性和渐进行为是判定能量动量赝张量合理性的重要依据之一[6],将以上结果代入式(12),可得:

$$\theta_0^0 = \frac{\gamma_\rho}{8\pi\rho}e^{2\gamma} = \frac{1}{8\pi\rho}\frac{t^2+\rho^2}{4\rho(t^2-\rho^2)}\frac{b\rho^{\frac{1}{2}}}{(t^2-\rho^2)^{\frac{1}{2}}} = \frac{b(t^2+\rho^2)}{32\pi[\rho(t^2-\rho^2)]^{\frac{3}{2}}}(t \geq \rho). \quad (14)$$

已知 $e^{2\gamma} = \frac{b\rho^{\frac{1}{2}}}{(t^2-\rho^2)^{\frac{1}{2}}} > 0$,所以有 $b > 0$,则 $\theta_0^0 > 0$,其正定性得到满足. 下面将对 θ_0^0 渐进行为进行讨论:

1) 在类时无穷大区域($t >> \rho$), $\theta_0^0 = \frac{b(t^2+\rho^2)}{32\pi[\rho(t^2-\rho^2)]^{\frac{3}{2}}} \to \frac{b(t^2+\rho^2)}{32\pi\rho^{\frac{3}{2}}t} \propto \frac{1}{t} \to 0$.

2) 在类光区域($t = \rho$) $\theta_0^0 \to +\infty$,即光锥处存在奇点.

由于柱面引力波的对称性,可以假设沿 x 轴传播,则有 $\rho = x$,结合式(11)、(12)可得: $\theta_0^1 = \frac{-\gamma_t x}{8\pi\rho^2}e^{2\gamma} = \frac{-\gamma_t\rho}{8\pi\rho^2}e^{2\gamma} = \frac{1}{8\pi\rho}\frac{t}{2(t^2-\rho^2)}\frac{b\rho^{\frac{1}{2}}}{(t^2-\rho^2)^{\frac{1}{2}}} = \frac{bt}{16\pi\rho^{\frac{1}{2}}} \cdot \frac{1}{(t^2-\rho^2)^{\frac{3}{2}}}$ 容易看出 θ_0^1 的正定性可以得到满足. 下面将对 θ_0^1 渐进行为进行讨论:

1) 在类时无穷大区域($t \geq \rho$),

$$\theta_0^1 = \frac{bt}{16\pi\rho^{\frac{1}{2}}}\frac{1}{(t^2-\rho^2)^{\frac{3}{2}}} \to \frac{b}{16\pi\rho^{\frac{1}{2}}t^2} \propto \frac{1}{t^2} \to 0.$$

2) 在类光区域($t = \rho$), $\theta_0^1 \to +\infty$,即光锥处存在奇点.

综合上面的分析和讨论不难发现,对于用 Rosen-virbhadra 表述计算得到的双极化态柱面引力波能量动量赝张量而言,在整个时空中的很大范围内,其能量、能量流的正定性和渐进行为都能得到很好的保证.

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Energy-momentum Pseudo-tensor of Cylindrical Gravitational Waves of Both the Polarized States Based on Rosen-virbhadra Expression

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Abstract: Using the expression of energy-momentum pseudo-tensor of cylindrical gravitational waves of both the polarized states, the authors got the expression of the energy and momentum after a great lot of calculations. The results are discussed, from these discussions, it is concluded that Cartesian coordinates is better than cylindrical coordinates to describe the cylindrical gravitational waves of both the polarized states.

Key words: Rosen-virbhadra; energy-momentum pseudo-tensor; both the polarized states; cylindrical gravitational waves

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Algorithm Discussion of High Precision for Wireless User Location

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Abstract: In recent years ,because of increasing mobile users ,the wireless location and business is the most attractive business in the world ,accurate location information to users has become one of criterion business for mobile communication ,so a discussed deeply wireless location technology is very value. This paper discusses it from high accuracy algorithm for wireless location on the network , by the emulation and comparison. It has shown that there are different algorithms in different surroundings for arriving at satisfied precision

Key words: wireless location; precision; emulation

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