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一类在无穷远点分支出十个极限环的多项式微分系统*

张　理[1,2],黄文韬[1]
(1. 桂林电子科技大学 计算科学与数学系,广西 桂林 541004;2. 安徽工业大学 数理学院,马鞍山 243002)

摘　要:运用一种间接的方法研究了一类七次系统在无穷远点的中心条件和极限环分支问题. 首先通过变换将原系统在无穷远点的极限环分支问题转化到在原点来研究,从而计算出该系统在原点的前98个奇点量,推导出原点成为中心和最高细焦点的条件,最后构造出在原点(即无穷远点)充分小的领域内分支出10个极限环的实例,首次证明了七次多项式系统在无穷远点能分支出10个极限环.

关键词:无穷远点;奇点量;焦点量;极限环分支
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在多项式微分自治系统定性理论中,无穷远点极限环分支问题是一个较困难的问题. 目前,多项式系统无穷远点极限环分支的研究主要集中在以下的阶实多项式微分系统

$$\left.\begin{aligned}\frac{dx}{dt} &= \sum_{k=0}^{2n} X_k(x,y) + (x\delta - y)(x^2+y^2)^n,\\ \frac{dy}{dt} &= \sum_{k=0}^{2n} Y_k(x,y) + (x + y\delta)(x^2+y^2)^n.\end{aligned}\right\}\quad (1)$$

其中 $X_k(x,y)$, $Y_k(x,y)$ 是关于 x,y 的 k 阶齐次多项式,n 是自然数. 对于 $n=1,2$ 时的一些特殊情形的系统在无穷远点极限环的研究已经取得了一定的成果,记 $I(m)$ 为 m 次多项式在无穷远点分支极限环的最大个数,迄今为止,$I(3)\geq 6$[2],$I(5)\geq 8$[3];但 $n=3$ 时的七次系统,研究成果非常少,目前最好的结果是 $I(7)\geq 9$[4],文中证明了一类七次系统在无穷远点能分支出10个极限环,即 $I(7)\geq 10$.

焦点量的计算是研究系统中心条件和极限环分支的一种有效的方法,文献[5]给出了一种新的计算焦点量的方法,即将实系统在原点的焦点量计算转化为其对应复伴随系统在原点奇点量的计算,这样就使高阶系统的焦点量的计算易于实现.

文中考虑的是下面一类七次多项式系统

$$\left.\begin{aligned}\frac{dx}{dt} &= X_1(x,y) + X_3(x,y) + (\delta x - y)(x^2+y^2)^3,\\ \frac{dy}{dt} &= Y_1(x,y) + Y_3(x,y) + (x - \delta y)(x^2+y^2)^3,\end{aligned}\right\}\quad (2)$$

这里 $X_k(x,y) = \sum_{i+j=k} A_{ij}x^iy^j$, $Y_k(x,y) = \sum_{i+j=k} B_{ij}x^iy^j$, A_{ij}, B_{ij}, $\delta\in\mathbb{R}$, $A_{12}\neq 0$, $k=1,3$.

1　奇点量与中心条件

系统(2)在复线性变换

$$z = x + iy,\ w = x - iy,\ \tau = it,\ i = \sqrt{-1}\quad (3)$$

化为一类复平面微分自治系统

$$\left.\begin{aligned}\frac{dz}{dt} &= a_{10}z + a_{01}w + a_{30}z^3 + a_{21}z^2w +\\ &\quad a_{12}zw^2 + a_{03}w^3 + (1 - i\delta)z^4w^3,\\ \frac{dw}{dt} &= -(b_{10}w + b_{01}z + b_{30}w^3 + b_{21}w^2z +\\ &\quad b_{12}zwz^2 + b_{03}z^3 + (1 + i\delta)w^4z^3.\end{aligned}\right\}\quad (4)$$

其中

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作者简介:张理(1980 -),男,安徽安庆人,桂林电子工业学院计算科学与数学系硕士研究生,主要从事微分方程定性理论方面的研究.

$$\left.\begin{aligned}
a_{30} &= (A_{03} + iA_{12} - A_{21} - iA_{30} + iB_{03} - B_{12} - iB_{21} + B_{30})/8,\\
a_{30} &= (A_{03} - iA_{12} - A_{21} + iA_{30} - iB_{03} - B_{12} + iB_{21} + B_{30})/8,\\
a_{03} &= (-A_{03} + iA_{12} + A_{21} - iA_{30} - iB_{03} - B_{12} + iB_{21} + B_{30})/8,\\
b_{03} &= (-A_{03} - iA_{12} + A_{21} + iA_{30} + iB_{03} - B_{12} - iB_{21} + B_{30})/8,\\
a_{21} &= (-3A_{03} - iA_{12} - A_{21} - 3iA_{30} - 3iB_{03} - B_{12} - iB_{21} + 3B_{30})/8,\\
b_{21} &= (-3A_{03} + iA_{12} - A_{21} + 3iA_{30} + 3iB_{03} + B_{12} + iB_{21} + 3B_{30})/8,\\
a_{12} &= (-3A_{03} - iA_{12} + A_{21} - 3iA_{30} + 3iB_{03} + B_{12} + iB_{21} + 3B_{30})/8,\\
b_{12} &= (-3A_{03} + iA_{12} + A_{21} + 3iA_{30} - 3iB_{03} + B_{12} - iB_{21} + 3B_{30})/8,\\
a_{01} &= (A_{01} - iA_{10} + iB_{01} + B_{10})/2,\\
b_{01} &= (A_{01} + iA_{10} - iB_{01} + B_{10})/2,\\
a_{10} &= (-A_{01} - iA_{10} - iB_{01} + B_{10})/2,\\
b_{10} &= (-A_{01} + iA_{10} + iB_{01} + B_{10})/2.
\end{aligned}\right\} \quad (5)$$

z,w 是复变量,且 $a_{12}b_{12} \neq 0$.

从式(5)可以看出系统(4)的系数满足共轭条件,即 $a_{ij} \neq b_{ij}$,称系统(2)和系统(4)互为伴随系统.

通过变换

$$z = \frac{\zeta}{(\zeta\eta)^4},\ w = \frac{\eta}{(\zeta\eta)^4},\ \mathrm{d}\tau = \frac{1}{(\zeta\eta)^3}\mathrm{d}T. \quad (6)$$

将系统(4)转化为下列系统

$$\begin{aligned}
\frac{\mathrm{d}\zeta}{\mathrm{d}T} &= (1 + \frac{1}{7}i\delta)\zeta + \frac{3}{7}a_{03}\zeta^{13}\eta^{16} + (\frac{3}{7}a_{12} + \frac{4}{7}b_{30})\zeta^{14}\eta^{15} + (\frac{3}{7}a_{21} + \frac{4}{7}b_{21})\zeta^{15}\eta_{14} + \\
&(\frac{3}{7}a_{30} + \frac{4}{7}b_{12})\zeta^{16}\eta^{13} + \frac{4}{7}b_{03}\zeta^{17}\eta^{12} + \frac{3}{7}a_{01}\zeta^{21}\eta^{22} + (\frac{3}{7}a_{10} + \frac{4}{7}b_{10})\zeta^{22}\eta^{21} + \frac{4}{7}b_{01}\zeta^{23}\eta^{20},\\
\frac{\mathrm{d}\eta}{\mathrm{d}T} &= -[(1 - \frac{1}{7}i\delta)\eta + \frac{3}{7}b_{03}\eta^{13}\zeta^{16} + (\frac{3}{7}b_{12} + \frac{4}{7}a_{30})\eta^{14}\zeta^{15} + (\frac{4}{7}a_{21} + \frac{3}{7}b_{21})\eta^{15}\zeta^{14} + \\
&(\frac{3}{7}b_{30} + \frac{4}{7}a_{12})\eta^{16}\zeta^{13} + \frac{4}{7}a_{03}\eta^{17}\zeta^{12} + \frac{3}{7}b^{01}\eta^{21}\zeta^{22} + (\frac{3}{7}b_{10} + \frac{4}{7}a_{10})\eta^{22}\zeta^{21} + \frac{4}{7}a_{01}\eta^{23}\zeta^{20}].
\end{aligned} \quad (7)$$

这样系统(4)在无穷远点的极限环分支问题就转化为系统(7)在原点的极限环分支问题,下面只要讨论系统(7)在原点的极限环分支情况即可.

根据计算奇点量的方法[6],利用计算机代数系统 Mathematica 对系统(7)$|_{\delta=0}$在原点的奇点量进行计算和化简有:

定理 1　系统(7)$|_{\delta=0}$无穷远点前 98 个奇点量如下:

$$\mu_7 = 0, \mu_{14} = \frac{1}{7}(b_{21} - a_{21}), \mu_{21} = \frac{1}{7}(b_{10} - a_{10}),$$

$$\mu_{28} = \frac{1}{7}(a_{12}a_{30} - b_{12}b_{30}),$$

$$\mu_{35} = \frac{1}{14}(3a_{01}a_{30} + a_{12}b_{01} - a_{01}b_{12} - 3b_{01}b_{30}),$$

(Ⅰ)若 $a_{30} = b_{12}/3, b_{30} = a_{12}/3$,则

$\mu_{42} = \mu_{49} = \mu_{56} = \mu_{63} = \mu_{70} = \mu_{84} = \mu_{91} = \mu_{98} = 0$;

(Ⅱ)若 $a_{30} \neq b_{12}/3, b_{30} \neq a_{12}/3$,则

$$\mu_{42} = \frac{1}{14}(-3a_{03}a_{30}^2 + a_{03}a_{30}b_{12} - a_{12}b_{03}b_{30} + 3b_{03}b_{30}^2,$$

$$\mu_{49} = \frac{1}{56}(-a_{01}a_{12}b_{03} + a_{03}b_{01}b_{12}),$$

$$\mu_{56} = -\frac{1}{56}(a_{12}^2b_{03} - a_{03}b_{12}^2(a_{21} + b_{21}),$$

$$\mu_{63} = -\frac{3}{112}(a_{12}^2b_{03} - a_{03}b_{12}^2)(a_{10} + b_{10}),$$

$$\mu_{70} = \frac{1}{336}(a_{12}^2b_{03} - a_{03}b_{12}^2)(a_{03}b_{03} - 4a_{12}b_{12}),$$

$$\mu_{77} = \mu_{84} = \mu_{91} = 0,$$

$$\mu_{98} = \frac{3}{8}a_{12}^2b_{12}^2(a_{12}^2b_{03} - a_{03}b_{12}^2),$$

当 $k \neq 7i, i < 14, i \in N$ 时,μ_k. 在上述 μ_k 的表达式中令 $\mu_1 = \mu_2 = \cdots\mu_{k-1} = 0, k = 2,3,\cdots,98$.

定理 2　系统(7)$|_{\delta=0}$原点的前 98 个奇点量均为零,当且仅当下列二组条件之一成立:

$$(i)\ a_{21} = b_{21}, a_{10} = b_{10}, b_{12} = 3a_{30}, a_{12} = 3b_{30}, \quad (8)$$

$$(ii)\ a_{21} = b_{21}, a_{10} = b_{10}, a_{12}a_{30} = b_{30}b_{12}, a_{12}b_{01} = a_{01}b_{12}, a_{12}^2b_{03} = b_{12}^2a_{03}, a_{12}b_{12} \neq 0. \quad (9)$$

证明　从定理 1 可以看出充分性是成立的,往证必要性.

由 $\mu_{28} = (a_{12}a_{30} - b_{12}b_{30})/7, a_{12}b_{12} \neq 0$,则存在常数 r,使得 $a_{30} = rb_{12}, b_{30} = ra_{12}$,代入 μ_{35} 的表达式得 $\mu_{35} = -(a_{12}b_{01} - a_{01}b_{12})(-1 + 3r)/14$,故 $r = 1/3$,则(i)式成立,否则存在另一常数 q,使得 $a_{01} = qa_{12}, b_{01} = qb_{12}$,往下不难推导出($ii$)式亦成立. 于是定理 2 得证.

文献[5]中定义了原点的基本 Lie - 不变量，证明了奇点量结构定理和广义对称原理. 为了得到系统(7)原点成为中心的和最高阶细焦点的条件，就必须找出系统(7)$|_{\delta=0}$的所有基本 Lie - 不变量，由文献[5]的方法有

引理 1 系统(7)$|_{\delta=0}$的基本 Lie - 不变量有 22 个：

$$\left.\begin{array}{l} a_{10},b_{10},a_{21},b_{21}, \\ a_{12}a_{30},b_{12}b_{30},a_{01}a_{30},b_{01}b_{30},a_{03}b_{03},a_{12}b_{12},a_{12}b_{01},a_{01}b_{12}, \\ a_{30}^2a_{03},b_{30}^2b_{03},b_{12}^2a_{03},a_{12}^2b_{03},a_{03}b_{12}a_{30},b_{03}a_{12}b_{30},a_{01}b_{03}a_{12},b_{01}a_{03}b_{12}. \end{array}\right\} \quad (10)$$

定理 3 系统(7)$|_{\delta=0}$原点所有的奇点量为零的充要条件是系统原点的前 98 个奇点量为零，即定理 2 的二条件之一成立. 因而定理 2 的二个条件即为系统(4)$|_{\delta=0}$原点的中心条件.

证明 必要性是显然的，下证充分性.

如果条件条件(i)成立，则系统(4)$|_{\delta=0}$是 Hamilton 系统. 若条件(ii)成立，从定理 2 及其证明的过程可知 $a_{30}=rb_{12},b_{30}=ra_{12},a_{01}=qa_{12},b_{01}=qb_{12}$，将其带入(10)式中并考虑(9)式其它的条件，可得系统(7)$|_{\delta=0}$的所有基本 Lie - 不变量是对称的，即系统(7)$|_{\delta=0}$满足广义对称原理[5]. 因此上述的任意条件成立均能导出系统(7)$|_{\delta=0}$原点的所有奇点量为零.

推论 1 系统(4)$|_{\delta=0}$的无穷远点为中心当且仅当定理 2 的二个条件之一成立.

2 焦点量与奇点量之间的关系

考虑实系统

$$\left.\begin{array}{l} \dfrac{d\zeta}{dT}=-\delta\zeta-\eta+\sum\limits_{k=2}^{\infty}X_k(\zeta,\eta)\ , \\ \dfrac{d\eta}{dT}=-\delta\zeta-\delta\eta+\sum\limits_{k=2}^{\infty}\Psi_k(\zeta,\eta)\ . \end{array}\right\} \quad (11)$$

在极坐标变换 $\zeta=\rho\cos\theta,\eta=\rho\sin\theta$ 下变为

$$\frac{d\rho}{d\theta}=\rho\frac{-\delta+\sum\limits_{k=2}^{\infty}\rho^{k-1}\varphi_{k+1}(\theta)}{1+\sum\limits_{k=2}^{\infty}\rho^{k-1}\phi_{k+1}(\theta)}\ , \quad (12)$$

其中

$\varphi_{k+1}(\theta)=\cos\theta X_k(\cos\theta,\sin\theta)+\sin\theta Y_k(\cos\theta,\sin\theta)$,

$\phi_{k+1}(\theta)=\cos\theta X_k(\cos\theta,\sin\theta)-\sin\theta Y_k(\cos\theta,\sin\theta),k=2,3,\cdots$.

对于充分小的 h，方程(11)满足初始条件 $\rho|_{\delta=0}=h$ 的解记为 $\rho=\rho(\theta,h)=\sum\limits_{m=1}^{\infty}v_m(\theta)h^m$，其中 $v_1(\theta)=e^{-\delta\theta},v_m(0)=0,m=2,3,\cdots$. 系统(2)在原点的第 0 个焦点量为 $v_1(2\pi)-1$，即 $e^{-2\pi\delta}-1$，第 k 个焦点量为 $v_{2k+1}(2\pi),k=1,2,\cdots$.

据文献[3，定理 2.3]有

引理 2 如果 $v_{2m+1}(2\pi)$是系统(11)$|_{\delta=0}$原点的第 m 个焦点量，μ_m 是系统(11)$|_{\delta=0}$的伴随系统原点的第 m 个奇点量，则

$$\mu_m \backsim \frac{1}{i\pi}v_{2m+1}(2\pi),m=1,2,\cdots . \quad (13)$$

从引理 2 可以看出系统(11)$|_{\delta=0}$在原点的焦点量与它伴随系统的奇点量之间的关系，由此系统(7)$|_{\delta=0}$的伴随系统焦点量的计算可转化为系统(7)$|_{\delta=0}$的奇点量的计算.

3 极限环分支

由定理 1 和定理 3 易得

定理 4 系统原点为 98 阶细奇点，即 $\mu_1=\mu_2=\cdots=\mu_{97}=0,\mu_{98}\neq0$ 的充要条件是

$$\left.\begin{array}{l} a_{10}=b_{10}=0,a_{01}=b_{01}=0, \\ a_{12}^2b_{03}\neq b_{12}^2a_{03},\delta=0, \\ a_{21}=b_{21}=0,a_{30}=b_{30}=0, \\ a_{03}b_{03}=4a_{12}b_{12},a_{12}b_{12},a_{12}b_{12}\neq0\ . \end{array}\right\} \quad (14)$$

且细奇点的最高阶数是 98 阶.

推论 2 系统(7)$|_{\delta=0}$的伴随系统原点为 98 阶细焦点当且仅当(14)式成立，且 98 阶是细焦点的最高阶数.

由定理 1 和推论 2，现构造出扰动系统(7)在原点分支出 10 个极限环的实例如下：

定理 5 如果系统(7)的系数满足

$$\left.\begin{array}{l} a_{21}=\varepsilon_3-i\varepsilon_9,b_{21}=\bar{a}_{21},a_{10}=-\varepsilon_2+i\varepsilon_8,b_{10}=\bar{a}_{10}, \\ a_{12}=b_{12}=1,\delta=-\varepsilon_{10}, \\ a_{30}=\varepsilon_5+i\varepsilon_7,b_{30}=\bar{a}_{30},a_{01}=-\varepsilon_4+i\varepsilon_6,b_{01}=\bar{a}_{01}, \\ a_{03}=2i-i\varepsilon_1,b_{03}=\bar{a}_{03}. \end{array}\right\} (15)$$

(相应地系统(2)的系数就由(5)式确定)，其中 ε_i 是满足 $0<\varepsilon_{10}\ll\varepsilon_9\ll\cdots\ll\varepsilon_1\ll1$ 的小参数，则系统(7)的伴随系统在原点充分小的领域内将产生 10 个极限环.

证明 由定理 1 及 $v_1(2\pi)-1=e^{-2\pi\delta}-1$ 与(15)式，经过细致的计算有

$$
\left.\begin{aligned}
&v_1(2\pi)-1=2\pi\varepsilon_{10}+o(\varepsilon_{10}),\\
&v_{29}(2\pi)=[-\frac{2}{7}\pi+\omega_1(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8,\varepsilon_9)]\varepsilon_9+o(\varepsilon_9),\\
&v_{43}(2\pi)=[\frac{2}{7}\pi+\omega_2(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8,\varepsilon_9)]\varepsilon_8+o(\varepsilon_8),\\
&v_{57}(2\pi)=[-\frac{2}{7}\pi+\omega_3(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8,\varepsilon_9)]\varepsilon_7+o(\varepsilon_7),\\
&v_{71}(2\pi)=[\frac{2}{7}\pi+\omega_4(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8,\varepsilon_9)]\varepsilon_6+o(\varepsilon_6),\\
&v_{85}(2\pi)=[-\frac{2}{7}\pi+\omega_5(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8,\varepsilon_9)]\varepsilon_5+o(\varepsilon_5),\\
&v_{99}(2\pi)=[\frac{2}{7}\pi+\omega_6(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8,\varepsilon_9)]\varepsilon_4+o(\varepsilon_4),\\
&v_{113}(2\pi)=[-\frac{2}{7}\pi+\omega_7(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8,\varepsilon_9)]\varepsilon_3+o(\varepsilon_3),\\
&v_{127}(2\pi)=[\frac{2}{7}\pi+\omega_8(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8,\varepsilon_9)]\varepsilon_2+o(\varepsilon_2),\\
&v_{141}(2\pi)=[-\frac{2}{7}\pi+\omega_1(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8,\varepsilon_9)]\varepsilon_1+o(\varepsilon_1),\\
&v_{197}(2\pi)=\frac{3}{20}\pi+o(1).
\end{aligned}\right\}\quad(16)
$$

其中 $\omega_i(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5,\varepsilon_6,\varepsilon_7,\varepsilon_8,\varepsilon_9)$ 在(0,0,0,0,0,0,0,0,0)解析且 $\omega_i(0,0,0,0,0,0,0,0,0)=0, i=1,2,\cdots,9$.

由(16)式得 $v_{14(m-1)+1}(2\pi)_{14m+1}(2\pi)<0$, $|v_{14(m-1)+1}(2\pi)|\ll|v_{14m+1}(2\pi)|$ $(m=1,3,4,5,6,7,8,9,10,14)$,根据经典的 Buatin 理论可知系统(7)的伴随系统在原点充分小的领域内经微扰至少可分支出 10 个极限环.

推论 3　系统(2)在无穷远点充分小的领域内经微扰可分支出 10 个极限环.

推论 3 首次证明了七次多项式系统在无穷远点分支出 10 个极限环的结果.

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Polynomial Differential System with Ten Limit Cycles at Infinity

ZHANG Li, *HUANG Wen-tao*

(1. Department of Computing Science and Mathematics, Guilin University of Electronic Technology, Guilin 541004, China; 2. School of Mathematics and Physics, Anhui University of Technology, Ma'anshan 243002, China)

Abstract: An indirect method is used to study bifurcations of limit cycles at infinity for a class of seventh-order polynomial differential system. First, the problem for bifurcations of limit cycles in the system at infinity is transformed into that at the origin. By the computation of fist 98 singular quantities, the conditions of the origin (correspondingly, infinity) to be the highest degree fine focus are derived. Finally, the system that bifurcates nine limit cycles in the neighborhood of infinity is constructed, which is proved that ten limit cycles can bifurcated at infinity for a class of seven-order polynomial system firstly.

Key words: infinity; singular quantity; focal value; bifurcation of limit cycles

(编辑　张小强)