Vatiational Methods for Anti-periodic Traveling Wave Solutions to a Forced Two-dimensional Generalized KdV Equation*

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ABSTRACT Using variational methods, we study the existence of the anti-periodic traveling wave solutions to a forced two-dimensional generalized KdV equation.

KEYWORDS variational methods / KdV equation; anti-periodic traveling wave solutions

0 Introduction

The two-dimensional KdV equation was first derived by Kadomtsev and Petviashvili in 1970[1], and it is also referred to as the KP equation. In [2], Aizicovici and Wen studied the existence and uniqueness of anti-periodic traveling wave solutions to a forced (inhomogeneous) generalized KP equation with the aid of monotonic method[3,4] and Schauder's fixed point theorem.

In this paper, we use the variational methods to study the existence of antiperiodic traveling wave solutions to the KP equation, we allow a broad class of functions $f(u)$ in the KP equation under investigation, in contrast with $f(u)$ being monotonically nondecreasing in [2].

1 Reduction of the Problem

We consider the generalized inhomogeneous two-dimensional KdV equation (see [1], [2], [5]):

$$\{u_t + [f(u)]_x + \alpha u_{xxx}\}_x + \beta u_{yy} + \bar{g} = 0, \qquad (t \geqslant 0, x, y \in R), \tag{1}$$

where $f \in C^2(R)$, $\alpha > 0$, and $\beta \neq 0$ are given constants, while $\bar{g}$ denotes a real-valued function of x, y and t.

We are interested in the existence of anti-periodic traveling wave solutions to Eq. (1), of the

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form

$$u(x,y,t)=U(z),z=ax+by-wt,$$

where $a>0$, and b,w are real constants, consequently, we make the natural assumption that $\tilde{g}$ depends on z only, i. e.

$$\tilde{g}(x,y,t)=g(ax+by-wt),\text{ with } g:R\to R.$$

Straightforward computations then show that (1) reduced to the fourth-order ordinary differential equation

$$U^{(4)}(z)-cU''(z)+\gamma\frac{d^2}{dz^2}f(U(z))+g_1(z)=0, \tag{2}$$

where

$$c=\alpha^{-4}(wa-\beta b^2),\gamma=\alpha^{-1}a^{-2},g_1(z)=\alpha^{-1}a^{-4}g(z). \tag{3}$$

We consider (2) in conjunction with the anti-periodic condition

$$U(z+T)=-U(z),z\in R, \tag{4}$$

where $T>0$ is fixed.

Let $f\in C^2(R)$ and $g_1\in C(R)$.

Define $G(z)$ such that $G''(z)=g_1(z)$.

It is easily to see that in order to prove the existence of the solution of (2) and (4), it is sufficient to prove the existence of the anti-periodic solution of the following second order ordinary differential equation

$$\left.\begin{aligned}&-\ddot{U}(z)+cU(z)+F(U(z))=G(z), && z\in R\\&U(z+T)=-U(z), && z\in R\end{aligned}\right\} \tag{5}$$

where

$$F(x)=-\gamma f(x),\quad x\in R.$$

More generally, we consider the following second-order anti-periodic boundary value problems:

$$\left.\begin{aligned}&-\ddot{x}+Ax+V'(x)=h(t),x\in R^n\\&x(0)=-x(T)\end{aligned}\right\} \tag{6}$$

where $A=(a_{ij})_{n\times n}$ is a symmetric matrix, $T>0$, $V\in C^1(R^n,R)$, $h\in L^2(R,R^n)$.

2 Variational Methods and Main Results

Consider the Sobolev space

$$H\equiv H^1([0,T],R^n) \tag{7}$$

On H we define the following inner product:

$$\ll x,y\gg=\int_0^T(\dot{x}\cdot\dot{y}+x\cdot y)dt,\forall\ x,y\in H.$$

The norm defined by $\ll \cdot , \cdot \gg$ on H is the usual H^1 norm:

$$\| x \|_H = (\int_0^T |\dot{x}|^2 dt + \int_0^T |x|^2 dt)^{\frac{1}{2}} \tag{8}$$

Define the subspace of H:

$$E = \{x \in H | x(0) = - x(T)\}, \tag{9}$$

and define functional $\mathcal{J}: E \to R$ as

$$\mathcal{J}(x) = \int_0^T \left(\frac{1}{2} |\dot{x}|^2 + \frac{1}{2} \langle Ax, x \rangle + V(x) - \langle h, x \rangle \right) dt \qquad \forall x \in E \tag{10}$$

Then it is easy to prove the following variational principle:

Lemma 1. The critical points of $\mathcal{J}$ in E is the solutions of (6).

Lemma 2. (Poincare'type inequality [4]) For all $x \in E$, we have

$$|x(t)| \leqslant \frac{1}{2} T^{\frac{1}{2}} (\int_0^T |\dot{x}(t)|^2 dt)^{\frac{1}{2}}, \qquad 0 \leqslant t \leqslant T. \tag{11}$$

Lemma 3. On E the usual H^1 norm is equivalent to the following norm $\| \ \|$:

$$\| x \| = (\int_0^T |\dot{x}|^2 dt)^{\frac{1}{2}}, \qquad \forall x \in E. \tag{12}$$

Proof. By Poincare'type inequality (Lemma 8), we have that for all $x \in E$

$$\int_0^T |x|^2 dt \leqslant T \max_{0 \leqslant t \leqslant T} |x(t)|^2 \leqslant \frac{T^2}{4} \int_0^T |\dot{x}|^2 dt, \tag{13}$$

$$\int_0^T |\dot{x}|^2 dt \leqslant \int_0^T (|\dot{x}|^2 + |x|^2) dt \leqslant \left(1 + \frac{T^2}{4}\right) \int_0^T |\dot{x}|^2 dt. \tag{14}$$

Lemma 4. $\mathcal{J}(x)$ is weakly lower semicontinuous, i. e. if $x_n \to x$ weakly in E then

$$\mathcal{J}(x) \leqslant \liminf_{n \to \infty} \mathcal{J}(x_n), \tag{15}$$

Proof. Suppose that $x_n \to x$ weakly in E, then

$$\int_0^T |\dot{x}|^2 dt \leqslant \liminf_{n \to \infty} \int_0^T |\dot{x}_n|^2 dt. \tag{16}$$

Moreover, by the Rellich-Kondrachov embedding theorem, we have that x_n has a subsequence, still denoted by x_n, such that $x_n \to x$ uniformly on $[0, T]$ and

$$\int_0^T\left(\frac{1}{2}\langle Ax_n,x_n\rangle+V(x_n)-\langle h,x_n\rangle\right)dt$$

$$\to\int_0^T\left(\frac{1}{2}\langle Ax,x\rangle+V(x)-\langle h,x\rangle\right)dt. \tag{17}$$

By (16),(17),it follows that

$$\mathcal{J}(x)\leqslant\liminf_{n\to\infty}\mathcal{J}(x_n)$$

Lemma 5. Considering system (6),suppose

$$\left.\begin{aligned}&\frac{1}{2}\langle Ax,x\rangle+V(x)\geqslant-\frac{\alpha}{2}|x|^2-M,\forall\ x\in R^n\\&1-\frac{\alpha}{4}T^2>0\end{aligned}\right\} \tag{18}$$

where α and M are two positive constants. Then

(i) $\mathcal{J}(x)$ is coercive in E,i. e. $\mathcal{J}(x)\to\infty$ whenever $\|x_n\|\to\infty$ as $n\to\infty$.

(ii) $\mathcal{J}(x)$ is bounded from below in E.

Proof. (i) By the assumption (18) and inequality (13),we have

$$\int_0^T\left(\frac{1}{2}\langle Ax,x\rangle+V(x)\right)dt\geqslant-\frac{\alpha}{2}\int_0^T|x|^2dt-MT\geqslant-\frac{\alpha T^2}{8}\|\dot{x}\|_{L^2}^2-MT. \tag{19}$$

By Hölder's inequality and inequality (13) we have

$$\int_0^T\langle h,x\rangle dt\leqslant\|h\|_{L^2}\cdot\|x\|_{L^2}\leqslant\frac{T}{2}\|h\|_{L^2}\cdot\|\dot{x}\|_{L^2} \tag{20}$$

By (10),(19),(20),we have

$$\mathcal{J}(x)\geqslant\left(\frac{1}{2}-\frac{\alpha}{8}T^2\right)\|\dot{x}\|_{L^2}^2-\frac{T}{2}\|h\|_{L^2}\cdot\|\dot{x}\|_{L^2}. \tag{21}$$

By $1-\frac{\alpha}{4}T^2>0$,it follows that $\mathcal{J}(x_n)\to\infty$ when $\|x_n\|_E=\|\dot{x}_n\|_{L^2}\to\infty$.

(ii) From (21),it is easy to prove that $\mathcal{J}$ is bounded from below.

Now we prove our main result:

Theorem 1 Suppose the condition (18) of Lemma 5 holds,then for any $0<T\leqslant\frac{2}{\sqrt{\alpha}}$,system (6) has at least an T—anti-periodic solution.

Proof. By Lemma 4 and 11,$\mathcal{J}(x)$ attains its infimum in E. Hence by the standard regularity theory,the minimum solution $x(t)$ is $C^2(R,R^n)$ solution of (6)[6,7].

Example 1. Assume symmetric matrix $A=(a_{ij})_{n\times n}$ is semipositive definite and $V(x)=\frac{1}{p}|x|^p$, $p\geqslant2$. Then the condition (18) holds for all $\alpha>0$. Therefore (6) has at least an T-anti-periodic solution by Theorem 1. Specially, considering system (5), if $c\geqslant 0, F(x)=|x|^{p-2}x, p\geqslant 2$, then (5) has at least an T-anti-periodic solution for any $T>0$.

Example 2. Considering system (5), let $c\geqslant 0$ and $F(x)=\cos x, x\in R$, then (5) has at least an T-anti-periodic solution for any $T>0$. In fact, in Theorem 1, let $n=1, A=c\geqslant 0, V(x)=\int_0^x \cos x dx=\sin x$. By $\sin x\geqslant -1$ for any $x\in R$, therefore condition (18) holds for all $a>0$.

Remark. In [2], the authors only consider the case that $F(x)$ must be monotonic. In our Example 2, $F(x)=\cos x$ is not monotonic in real line.

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受迫二维广义 KdV 方程的反周期行波解的变分方法

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摘 要 用变分法研究了受迫二维广义 KdV 方程的反周期行波解的存在性。

关键词 变分法 / KdV 方程；反周期行波解

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